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Stroh-like formalism for the coupled stretching–bending analysis of composite laminates

Chyanbin Hwu *

Institute of Aeronautics and Astronautics, National Cheng Kung University, Tainan 70101, Taiwan, ROC

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Abstract

Due to the two-dimensional nature of thin plates, the lamination theory considering the composite laminates with in-plane and plate bending problems coupling each other is treated in this paper by using complex variable formulation. By following the steps of Stroh formalism for two-dimensional linear anisotropic elasticity, a displacement complex variable formalism developed by the other researchers was introduced and re-derived in a different but more Stroh-like way. In addition, a brand-new mixed formalism (mixed use of displacements and stresses as basic functions) is established to compensate the displacement formalism. In order to transfer all the related formulae and mathematical techniques of the Stroh formalism to these two formalisms, the general solutions for the basic equations of lamination theory and their associated eigenrelations have been purposely arranged in the form of Stroh formalism. Moreover, by using the presently developed mixed formalism, the explicit expressions for the fundamental matrix and eigenvectors are obtained first time for the most general composite laminates. Furthermore, letting the coupling stiffness vanish, the formalism has been reduced to the case of symmetric laminates and checked by a recently developed Stroh-like formalism for the plate bending problems. The comparison between Stroh formalism for two-dimensional problem, Stroh-like formalism for plate bending problem, displacement formalism and mixed formalism is then made at the end of this paper, and through their connection some useful relations are obtained.

Keywords: Lamination theory; Complex variable formalism; Anisotropic plates; Stroh formalism; Lekhnitskii formalism; Plate bending problem; Inplane problem

1. Introduction

Although the classical lamination theory was developed long time ago (Jones, 1974), it is not easy to apply this theory to find an analytical solution for the problem with curvilinear boundaries, especially when the laminates are composed of the laminae that will make the in-plane and plate bending problems couple

* Fax: +886-6-238-9940.
E-mail address: chwu@mail.ncku.edu.tw (C. Hwu).

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each other. For example, the problems of composite laminates with holes/cracks/inclusions, which have been discussed and solved vastly in two-dimensional problems, are still difficult to be solved when the laminates are subjected to out-of-plane bending moments. Because this kind of problems was solved by the complex variable formalism in two-dimensional deformation, it is hoped that similar formalism can be developed for the classical lamination theory.

For two-dimensional linear anisotropic elasticity, there are two major complex variable formalisms in the literature. One is the Lekhnitskii formalism (Lekhnitskii, 1963, 1968) which starts with the equilibrated stress functions followed by constitutive laws, strain–displacement relation and compatibility equations; the other is the Stroh formalism (Stroh, 1958, 1962) which starts with the compatible displacements followed by strain–displacement relation, constitutive laws and equilibrium equations. Due to the special feature that Stroh formalism possesses an eigenrelation which relates the eigenmodes of stress functions and displacements to the material properties, recently Stroh formalism becomes more attractive than Lekhnitskii formalism, especially when the book (Ting, 1996) emphasized on Stroh formalism was published. Due to this reason, the efforts of this paper will be focused upon the establishment of a counterpart of Stroh formalism for the lamination theory. If the formulae developed for the lamination theory can be purposely arranged into the form of Stroh formalism for two-dimensional linear anisotropic elasticity, almost all the mathematical techniques developed for two-dimensional problems can be transferred to the coupled stretching–bending problems. Thus, by simple analogy, many unsolved lamination problems can now be solved if their corresponding two-dimensional problems have been solved successfully.

Tracing the literature, we found that Lekhnitskii has ever developed a complex variable formalism for the plate bending problems (Lekhnitskii, 1938), and used his formalism to solve the problems of orthotropic plates containing circular holes or rigid inclusions (Lekhnitskii, 1968). After that, very few contributions can be found in the literature for the improvement of complex variable formulation in plate bending problems, although some related works have been touched such as Qin et al. (1991). Recently, through the understanding of the connection between Stroh formalism and Lekhnitskii formalism for the two-dimensional problems, I (Hwu, in press) developed a Stroh-like formalism for the bending theory of anisotropic plates, which can be applied directly to the symmetric laminates. By this newly developed formalism, we successfully obtained the analytical solutions for the problem of anisotropic plates with holes/cracks/inclusions subjected to out-of-plane bending moments (Hsieh and Hwu, 2002a). However, it is difficult to apply the same approach to the unsymmetric laminates with in-plane and plate bending coupling. Instead of using the advantages of Stroh–Lekhnitskii's connection, a displacement-based derivation has been introduced by Lu and Mahrenholtz (1994) and modified by Cheng and Reddy (2002). In addition, some researchers devoted their efforts to the development and application of the complex variable method on the laminates with bending extension coupling such as (Becker, 1991; Zakharov, 1992). Due to the complexity, the resemblance between the Stroh formalism and the displacement formalism as well as the published complex variable methods is not perfect enough to employ most of the key features of Stroh formalism. Thus, to get the solutions for the bending extension coupling problems one cannot directly duplicate from their corresponding two-dimensional counterparts, detailed derivations instead are still needed such as a number of problems solved by Becker and Zakharov (Becker, 1992, 1993, 1995; Zakharov and Becker, 2000; Engels and Becker, 2002).

By comparing the formalisms developed by Hwu (in press), Lu and Mahrenholtz (1994) and Cheng and Reddy (2002), we found there may exist an alternative formalism that is more alike to the Stroh formalism for two-dimensional problems. In this paper, we call it mixed formalism because in our derivation the basic functions are not pure displacements or pure stresses but in-plane displacements plus plate bending moments. The associated eigenrelation of the mixed formalism shows that it is really more alike to the Stroh formalism. Moreover, by the mixed formalism, the explicit expressions of the material eigenvectors have been obtained in this paper. In addition, the explicit expressions of the fundamental elasticity matrices of Stroh-like formalism for symmetric/unsymmetric laminates have also been obtained through the use of mixed formalism developed in this paper (Hsieh and Hwu, 2002b).

To have a deep insight of the complex variable formalism for lamination theory, both the displacement and mixed formalisms are presented in this paper. By comparing these two formalisms with the Stroh formalism for two-dimensional problems (Ting, 1996) and the Stroh-like formalism for plate bending problems (Hwu, in press), we find that both of them are not perfectly match with the Stroh formalism. Combining use of these two formalisms may be a good approach for solving practical lamination problems. Thus, relations between these two formalisms are also provided in this paper.

2. Lamination theory

To describe the overall properties and mechanical behavior of a laminate, the most popular way is the classical lamination theory (Jones, 1974). According to the observation of actual mechanical behavior of laminates, Kirchhoff's assumptions are usually made in this theory. Based upon the Kirchhoff's assumptions, the displacement fields, the strain–displacement relations, the constitutive laws and the equilibrium equations can be written as follows.

2.1. Displacement fields

$$\begin{aligned} u(x,y,z) &= u_0(x,y) - z\frac{\partial w(x,y)}{\partial x}, \\ v(x,y,z) &= v_0(x,y) - z\frac{\partial w(x,y)}{\partial y}, \\ w(x,y,z) &= w_0(x,y), \end{aligned} \tag{2.1}$$

where u, v and w are the displacements in x, y and z directions, and u_0, v_0 and w_0 are the middle surface displacements.

2.2. Strain–displacement relations

$$\varepsilon_x = \varepsilon_x^0 + z\kappa_x, \qquad \varepsilon_y = \varepsilon_y^0 + z\kappa_y, \qquad \gamma_{xy} = \gamma_{xy}^0 + z\kappa_{xy}, \tag{2.2}$$

where

$$\begin{aligned} &\varepsilon_x^0 = \frac{\partial u_0}{\partial x}, \qquad \varepsilon_y^0 = \frac{\partial v_0}{\partial y}, \qquad \gamma_{xy}^0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}, \\ &\kappa_x = -\frac{\partial^2 w}{\partial x^2}, \qquad \kappa_y = -\frac{\partial^2 w}{\partial y^2}, \qquad \kappa_{xy} = -2\frac{\partial^2 w}{\partial x \partial y}, \end{aligned} \tag{2.3}$$

where $(\varepsilon_x, \varepsilon_y, \gamma_{xy})$ are the strains, $(\varepsilon_x^0, \varepsilon_y^0, \gamma_{xy}^0)$ are the mid-plane strains and $(\kappa_x, \kappa_y, \kappa_{xy})$ are the plate curvatures.

2.3. Constitutive laws

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}, \tag{2.4}$$

where N_x, N_y, N_{xy} are the resultant forces and M_x, M_y, M_{xy} are the resultant moments. A_{ij}, B_{ij} and D_{ij} are, respectively, the *extensional*, *coupling* and *bending stiffnesses*, and are determined by

$$A_{ij} = \sum_{k=1}^{n} (\overline{Q}_{ij})_k (h_k - h_{k-1}), \quad B_{ij} = \frac{1}{2}\sum_{k=1}^{n} (\overline{Q}_{ij})_k (h_k^2 - h_{k-1}^2), \quad D_{ij} = \frac{1}{3}\sum_{k=1}^{n} (\overline{Q}_{ij})_k (h_k^3 - h_{k-1}^3), \tag{2.5}$$

where h_k and h_{k-1} denotes, respectively, the location of the bottom and top surface of the kth lamina (Fig. 1). $(\overline{Q}_{ij})_k$ is the transformed stiffness matrix of the kth lamina.

2.4. Equilibrium equations

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} = 0, \qquad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0, \tag{2.6a}$$

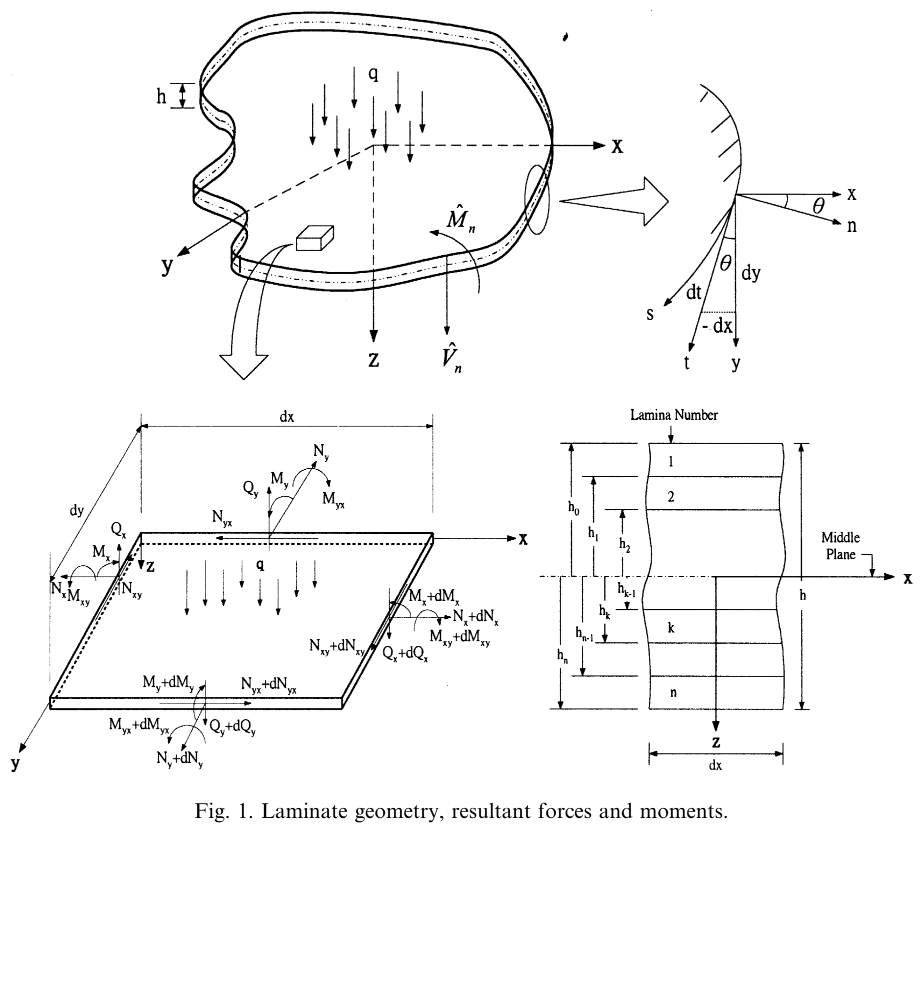

Fig. 1. Laminate geometry, resultant forces and moments.

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0, \tag{2.6b}$$

where q is the lateral distributed load applied on the laminates. Note that Eq. (2.6b) represents the forces equilibrium in the thickness direction, which is usually written in terms of the transverse forces Q_x and Q_y as

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0. \tag{2.7}$$

The moment equilibrium in the x- and y-directions shows that the transverse shear forces are related to the bending moments by

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}, \qquad Q_y = \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y}. \tag{2.8}$$

Substituting (2.8) into (2.7), we get Eq. (2.6b).

2.5. Governing equations

To get governing equations satisfying all the basic equations, we first use (2.3) to express the mid-plane strains ε_x^0, ε_y^0, γ_{xy}^0 and curvatures κ_x, κ_y, κ_{xy} in terms of the mid-plane displacements u_0, v_0 and w, then use (2.4) to express the resultant forces N_x, N_y, N_{xy} and moments M_x, M_y, M_{xy} in terms of the mid-plane displacements. After these direct substitutions, the three equilibrium equations (2.6) can now be written in terms of three unknown displacement functions u_0, v_0 and w as

$$\begin{aligned} &A_{11}\frac{\partial^2 u_0}{\partial x^2} + 2A_{16}\frac{\partial^2 u_0}{\partial x\partial y} + A_{66}\frac{\partial^2 u_0}{\partial y^2} + A_{16}\frac{\partial^2 v_0}{\partial x^2} + (A_{12}+A_{66})\frac{\partial^2 v_0}{\partial x\partial y} + A_{26}\frac{\partial^2 v_0}{\partial y^2} - B_{11}\frac{\partial^3 w}{\partial x^3} - 3B_{16}\frac{\partial^3 w}{\partial x^2\partial y} \\ &\quad - (B_{12}+2B_{66})\frac{\partial^3 w}{\partial x\partial^2 y} - B_{26}\frac{\partial^3 w}{\partial y^3} = 0, \end{aligned} \tag{2.9a}$$

$$\begin{aligned} &A_{16}\frac{\partial^2 u_0}{\partial x^2} + (A_{12}+A_{66})\frac{\partial^2 u_0}{\partial x\partial y} + A_{26}\frac{\partial^2 u_0}{\partial y^2} + A_{66}\frac{\partial^2 v_0}{\partial x^2} + 2A_{26}\frac{\partial^2 v_0}{\partial x\partial y} + A_{22}\frac{\partial^2 v_0}{\partial y^2} - B_{16}\frac{\partial^3 w}{\partial x^3} \\ &\quad - (B_{12}+2B_{66})\frac{\partial^3 w}{\partial x^2\partial y} - 3B_{26}\frac{\partial^3 w}{\partial x\partial y^2} - B_{22}\frac{\partial^3 w}{\partial y^3} = 0, \end{aligned} \tag{2.9b}$$

$$\begin{aligned} &D_{11}\frac{\partial^4 w}{\partial x^4} + 4D_{16}\frac{\partial^4 w}{\partial x^3\partial y} + 2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2\partial y^2} + 4D_{26}\frac{\partial^4 w}{\partial x\partial y^3} + D_{22}\frac{\partial^4 w}{\partial y^4} - B_{11}\frac{\partial^3 u_0}{\partial x^3} - 3B_{16}\frac{\partial^3 u_0}{\partial x^2\partial y} \\ &\quad - (B_{12}+2B_{66})\frac{\partial^3 u_0}{\partial x\partial y^2} - B_{26}\frac{\partial^3 u_0}{\partial y^3} - B_{16}\frac{\partial^3 v_0}{\partial x^3} - (B_{12}+2B_{66})\frac{\partial^3 v_0}{\partial x^2\partial y} - 3B_{26}\frac{\partial^3 v_0}{\partial x\partial y^2} - B_{22}\frac{\partial^3 v_0}{\partial y^3} = q, \end{aligned} \tag{2.9c}$$

which are the governing equations for the laminated plates.

The governing equations shown in (2.9a,b,c) are system of partial differential equations with three unknown functions u_0, v_0 and w. Due to the mathematical complexity of these equations, it is not easy to get solutions by solving these partial differential equations. In practical engineering applications, it is common to have a symmetric laminate or to construct a balanced laminate. In those cases the coupling stiffness components like B_{ij} and/or A_{16}, A_{26} and/or D_{16}, D_{26} will be zero, and Eqs. (2.9a,b,c) will be drastically simplified.

2.6. Boundary conditions

For the general cases of laminated plates, the in-plane and bending problems will couple each other. Hence, every boundary of the plates should be described by four prescribed values. Two of them correspond to the in-plane problems and the other two correspond to the bending problems. Generally, they may be expressed as

$$\begin{aligned} &u_n = \hat{u}_n \quad \text{or} \quad N_n = \widehat{N}_n \quad \text{or} \quad N_n = k_n u_n, \\ &u_t = \hat{u}_t \quad \text{or} \quad N_{nt} = \widehat{N}_{nt} \quad \text{or} \quad N_{nt} = k_t u_t, \\ &w_{,n} = \hat{w}_{,n} \quad \text{or} \quad M_n = \widehat{M}_n \quad \text{or} \quad M_n = k_m w_{,n}, \\ &w = \hat{w} \quad \text{or} \quad V_n = \widehat{V}_n \quad \text{or} \quad V_n = k_v w, \end{aligned} \tag{2.10}$$

where V_n is the well known *Kirchhoff force* of classical plate theory, or called *effective shear force* defined by

$$V_n = Q_n + \frac{\partial M_{nt}}{\partial t}. \tag{2.11}$$

The subscripts n and t denote, respectively, the directions normal and tangent to the boundary. The overhat denotes the prescribed value. The values in the n–t coordinate can be calculated from the values in the x–y coordinate according to the transformation laws.

3. Displacement formalism

The governing equations (2.9a,b,c) involve both in-plane and plate bending problems, i.e., these two problems are coupled each other if the coupling stiffnesses B_{ij} are not equal to zero. Due to the mathematical complexity, very few systematic approaches can be found in the literature. Because of the two-dimensional nature of the plate bending problems, it is hoped that the complex variable method which is powerful and elegant for the two-dimensional problems can also be applied to the plate bending problems. For the two-dimensional linear anisotropic elasticity, there are two important books published in the literature. One is by Lekhnitskii (1963), and the other is by Ting (1996). The former concerns a stress formalism which is generally called Lekhnitskii formalism, whereas the latter concerns a displacement formalism which is generally called Stroh formalism (Stroh, 1958). In this section, we will use the mid-plane displacements and slopes as our basic functions, which is therefore called displacement formalism. Actually, this approach has been introduced by Lu and Mahrenholtz (1994) and modified by Cheng and Reddy (2002). In addition to the displacement formalism, by using mid-plane displacements and bending moments as basic functions, a brand-new formalism called the mixed formalism will be introduced in the next section.

Although several different kinds of displacement formalisms have been developed in the literature, some fail in their eigenrelation and some fail in their complexity. To have a correct and clear formalism, in this section we re-derive the displacement formalism in a more Stroh-like approach. For the convenience of later derivation, we will firstly rewrite all the basic equations (2.1)–(2.8) in terms of tensor notation as follows:

$$\begin{aligned} &U_i = u_i + z\beta_i, \quad \beta_1 = -w_{,1}, \quad \beta_2 = -w_{,2}, \\ &\xi_{ij} = \varepsilon_{ij} + z\kappa_{ij} = \tfrac{1}{2}(U_{i,j} + U_{j,i}), \quad \varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \quad \kappa_{ij} = \tfrac{1}{2}(\beta_{i,j} + \beta_{j,i}), \\ &N_{ij} = A_{ijkl}\varepsilon_{kl} + B_{ijkl}\kappa_{kl}, \quad M_{ij} = B_{ijkl}\varepsilon_{kl} + D_{ijkl}\kappa_{kl}, \\ &N_{ij,j} = 0, \quad M_{ij,ij} + q = 0, \quad Q_i = M_{ij,j}, \qquad i,j,k,l = 1,2. \end{aligned} \tag{3.1}$$

Note that in the above tensor notation, we have made a slight change for some symbols and employed the following conventional replacements,

$$x \leftrightarrow 1, \quad y \leftrightarrow 2, \quad 11 \leftrightarrow 1, \quad 22 \leftrightarrow 2, \quad 12 \text{ or } 21 \leftrightarrow 6, \tag{3.2a}$$

for example,

$$\begin{aligned}
&u \leftrightarrow U_1, \quad v \leftrightarrow U_2, \quad u_0 \leftrightarrow u_1, \quad v_0 \leftrightarrow u_2, \\
&\varepsilon_x \leftrightarrow \xi_{11}, \quad \gamma_{xy} \leftrightarrow 2\xi_{12}, \quad \varepsilon_x^0 \leftrightarrow \varepsilon_{11}, \quad \gamma_{xy}^0 \leftrightarrow 2\varepsilon_{12}, \quad \kappa_x \leftrightarrow \kappa_{11}, \quad \kappa_{xy} \leftrightarrow 2\kappa_{12}, \\
&N_y \leftrightarrow N_{22}, \quad M_{xy} \leftrightarrow M_{12}, \quad Q_x \leftrightarrow Q_1, \\
&A_{1121} \leftrightarrow A_{16}, \quad B_{2211} \leftrightarrow B_{21}, \quad D_{1212} \leftrightarrow D_{66}, \ldots, \text{ etc.}
\end{aligned} \tag{3.2b}$$

It should be noted that the replacements of shear strain γ_{xy} and curvature κ_{xy} are not only symbol change but also two times difference, which is the same as the conventional contracted notation (Sokolkoff, 1956).

Substituting the strains/curvatures and displacements/slopes relations into the constitutive laws, i.e., substituting $(3.1)_2$ into $(3.1)_3$, the resultant forces and moments may be expressed in terms of mid-plane displacements u_i and slopes β_i as

$$N_{ij} = A_{ijkl}u_{k,l} + B_{ijkl}\beta_{k,l}, \qquad M_{ij} = B_{ijkl}u_{k,l} + D_{ijkl}\beta_{k,l}. \tag{3.3}$$

By employing the results of (3.3) to the equilibrium equations $(3.1)_4$, the governing equations may also be expressed in terms of the mid-plane displacements u_i and slopes β_i as

$$A_{ijkl}u_{k,lj} + B_{ijkl}\beta_{k,lj} = 0, \qquad B_{ijkl}u_{k,lij} + D_{ijkl}\beta_{k,lij} + q = 0. \tag{3.4}$$

Consider the homogeneous case that no lateral load is applied on the laminates, i.e., $q = 0$. Because the mid-plane displacements u_i and slopes β_i depend only on two variables, x_1 and x_2, and (3.4) are homogeneous partial differential equations, we may let

$$u_k = a_k^u f(z), \quad \beta_k = a_k^\beta f(z), \quad z = x_1 + \mu x_2, \qquad k = 1, 2. \tag{3.5}$$

Differentiation of (3.5) with respect to x_l gives

$$u_{k,l} = a_k^u(\delta_{l1} + \mu\delta_{l2})f'(z), \qquad \beta_{k,l} = a_k^\beta(\delta_{l1} + \mu\delta_{l2})f'(z), \tag{3.6}$$

in which the prime (′) denotes differentiation with respect to the argument z and δ_{ij} is Kronecker delta. Further differentiating with respect to x_j and x_i, we find that (3.4) with $q = 0$ will be satisfied if

$$\begin{aligned}
&\{\mathbf{Q}_A + \mu(\mathbf{R}_A + \mathbf{R}_A^{\mathrm{T}}) + \mu^2\mathbf{T}_A\}\mathbf{a}_u + \{\mathbf{Q}_B + \mu(\mathbf{R}_B + \mathbf{R}_B^{\mathrm{T}}) + \mu^2\mathbf{T}_B\}\mathbf{a}_\beta = \mathbf{0}, \\
&\boldsymbol{\mu}^{*\mathrm{T}}\{\mathbf{Q}_B + \mu(\mathbf{R}_B + \mathbf{R}_B^{\mathrm{T}}) + \mu^2\mathbf{T}_B\}\mathbf{a}_u + \boldsymbol{\mu}^{*\mathrm{T}}\{\mathbf{Q}_D + \mu(\mathbf{R}_D + \mathbf{R}_D^{\mathrm{T}}) + \mu^2\mathbf{T}_D\}\mathbf{a}_\beta = 0,
\end{aligned} \tag{3.7a}$$

where

$$\begin{aligned}
&\mathbf{Q}_A = A_{i1k1}, \qquad \mathbf{Q}_B = B_{i1k1}, \qquad \mathbf{Q}_D = D_{i1k1}, \\
&\mathbf{R}_A = A_{i1k2}, \qquad \mathbf{R}_B = B_{i1k2}, \qquad \mathbf{R}_D = D_{i1k2}, \\
&\mathbf{T}_A = A_{i2k2}, \qquad \mathbf{T}_B = B_{i2k2}, \qquad \mathbf{T}_D = D_{i2k2}, \\
&\mathbf{a}_u = \begin{Bmatrix} a_1^u \\ a_2^u \end{Bmatrix}, \quad \mathbf{a}_\beta = \begin{Bmatrix} a_1^\beta \\ a_2^\beta \end{Bmatrix}, \quad \boldsymbol{\mu}^* = \begin{Bmatrix} 1 \\ \mu \end{Bmatrix}.
\end{aligned} \tag{3.7b}$$

From the second and third equations of $(3.1)_1$ and the assumption of the slope β_k given in the second equation of (3.5), we get

$$a_2^\beta = \mu a_1^\beta. \tag{3.8}$$

Eqs. (3.7) and (3.8) constitute four equations with four unknowns a_1^u, a_2^u, a_1^β, a_2^β. Thus, the problem is solved in principle. Substituting (3.6) into (3.3), we have

$$\begin{aligned} N_{i1} &= -\mu \mathbf{b}_u f'(z), \qquad N_{i2} = \mathbf{b}_u f'(z), \\ M_{i1} &= -\mu \mathbf{d}^* f'(z), \qquad M_{i2} = \mathbf{d} f'(z), \end{aligned} \tag{3.9a}$$

where

$$\begin{aligned} \mathbf{b}_u &= (\mathbf{R}_A^{\mathrm{T}} + \mu \mathbf{T}_A)\mathbf{a}_u + (\mathbf{R}_B^{\mathrm{T}} + \mu \mathbf{T}_B)\mathbf{a}_\beta = -\frac{1}{\mu}\{(\mathbf{Q}_A + \mu \mathbf{R}_A)\mathbf{a}_u + (\mathbf{Q}_B + \mu \mathbf{R}_B)\mathbf{a}_\beta\}, \\ \mathbf{d} &= (\mathbf{R}_B^{\mathrm{T}} + \mu \mathbf{T}_B)\mathbf{a}_u + (\mathbf{R}_D^{\mathrm{T}} + \mu \mathbf{T}_D)\mathbf{a}_\beta, \\ \mathbf{d}^* &= -\frac{1}{\mu}\{(\mathbf{Q}_B + \mu \mathbf{R}_B)\mathbf{a}_u + (\mathbf{Q}_D + \mu \mathbf{R}_D)\mathbf{a}_\beta\}. \end{aligned} \tag{3.9b}$$

Note that the second equality of $(3.9\mathrm{b})_1$ comes from $(3.7\mathrm{a})_1$. Using the relation for the bending moments and transverse shear forces given in the third equation of $(3.1)_4$, and the definition for the effective transverse shear force given in (2.11), and the results for the bending moments given in $(3.9\mathrm{a})_2$, we obtain

$$\begin{aligned} Q_i &= \mu(\mathbf{d} - \mathbf{d}^*) f''(z), \\ V_1 &= \mu \mathbf{i}_1^{\mathrm{T}}(2\mathbf{d} - \mathbf{d}^*) f''(z), \qquad V_2 = \mu \mathbf{i}_2^{\mathrm{T}}(\mathbf{d} - 2\mathbf{d}^*) f''(z), \end{aligned} \tag{3.10a}$$

where

$$\mathbf{i}_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \qquad \mathbf{i}_2 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}. \tag{3.10b}$$

With the definitions of $\mathbf{d}$ and $\mathbf{d}^*$ given in (3.9b), the second equation of (3.7a) leads to

$$\boldsymbol{\mu}^{*\mathrm{T}}(\mathbf{d} - \mathbf{d}^*) = 0. \tag{3.11}$$

Substituting $(3.9\mathrm{a})_2$ into the symmetry condition of the twist moments, i.e., $M_{12} = M_{21}$, we have

$$d_1 = -\mu d_2^*. \tag{3.12}$$

Combining (3.11) and (3.12), we may express $\mathbf{d}^*$ in terms of $\mathbf{d}$, or vice versa. Through their relation, we now introduce a new vector $\mathbf{b}_\beta$ as

$$\mathbf{b}_\beta = \mathbf{d} + b_0 \mathbf{i}_1 = \mathbf{d}^* + \frac{b_0}{\mu}\mathbf{i}_2, \tag{3.13a}$$

where

$$b_0 = \tfrac{1}{2}\boldsymbol{\mu}^{*\mathrm{T}}\mathbf{b}_\beta = \tfrac{1}{2}(b_1^\beta + \mu b_2^\beta) = d_1 + \mu d_2. \tag{3.13b}$$

By the relation given in (3.13), the expressions for the bending moments and transverse shear forces obtained in $(3.9\mathrm{a})_2$ and (3.10a) can now be written as

$$\begin{aligned} M_{i1} &= (-\mu \mathbf{b}_\beta + b_0 \mathbf{i}_2) f'(z), \qquad M_{i2} = (\mathbf{b}_\beta - b_0 \mathbf{i}_1) f'(z), \\ Q_i &= b_0 \begin{Bmatrix} -\mu \\ 1 \end{Bmatrix} f''(z), \qquad V_i = \begin{Bmatrix} -\mu^2 b_2^\beta \\ b_1^\beta \end{Bmatrix} f''(z). \end{aligned} \tag{3.14}$$

Observing the results obtained in (3.9a) and (3.14), we introduce two stress functions

$$\phi_i = b_i^u f(z), \qquad \psi_i = b_i^\beta f(z). \tag{3.15}$$

With the use of these two stress functions, the moments, transverse shear forces and effective transverse shear forces can be expressed as

$$
\begin{aligned}
&N_{i1} = -\phi_{i,2}, \qquad N_{i2} = \phi_{i,1}, \\
&M_{i1} = -\psi_{i,2} - \tfrac{1}{2}\lambda_{i1}\psi_{k,k}, \qquad M_{i2} = \psi_{i,1} - \tfrac{1}{2}\lambda_{i2}\psi_{k,k}, \\
&Q_1 = -\tfrac{1}{2}\psi_{k,k2}, \qquad Q_2 = \tfrac{1}{2}\psi_{k,k1}, \\
&V_1 = -\psi_{2,22}, \qquad V_2 = \psi_{1,11},
\end{aligned} \tag{3.16}
$$

where λ_{ij} is the permutation tensor defined as

$$\lambda_{11} = \lambda_{22} = 0, \qquad \lambda_{12} = -\lambda_{21} = 1. \tag{3.17}$$

Up to now, the formalism is almost complete because the displacements, slopes, moments, and transverse shear forces have all been expressed elegantly in (3.5), (3.15) and (3.16). The eigenvalues μ and the displacement eigenvectors $\mathbf{a}_u$, $\mathbf{a}_\beta$ can be obtained from (3.7a) and (3.8), and the stress function eigenvectors $\mathbf{b}_u$, $\mathbf{b}_\beta$ can be obtained from (3.9b) and (3.13). From (3.7) and (3.8), the determination of the eigenvalues μ will lead to an equation of 8th order polynomial, which can be proved to have eight roots with four pairs of complex conjugates (Cheng and Reddy, 2002). By arranging the complex eigenvalues whose imaginary parts are positive to be the first four eigenvalues, and superimposing all their corresponding solutions, the solutions shown in (3.5) and (3.15) may now be written in a compact matrix form as

$$\mathbf{u}_d = 2\mathrm{Re}\{\mathbf{A}_d\mathbf{f}(z)\}, \qquad \boldsymbol{\phi}_d = 2\mathrm{Re}\{\mathbf{B}_d\mathbf{f}(z)\}, \tag{3.18a}$$

where

$$\mathbf{u}_d = \begin{Bmatrix} \mathbf{u} \\ \boldsymbol{\beta} \end{Bmatrix}, \qquad \boldsymbol{\phi}_d = \begin{Bmatrix} \boldsymbol{\phi} \\ \boldsymbol{\psi} \end{Bmatrix}, \tag{3.18b}$$

$$
\begin{aligned}
&\mathbf{A}_d = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4], \qquad \mathbf{B}_d = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{b}_4], \\
&\mathbf{f}(z) = \begin{Bmatrix} f_1(z_1) \\ f_2(z_2) \\ f_3(z_3) \\ f_4(z_4) \end{Bmatrix}, \quad z_k = x_1 + \mu_k x_2, \qquad k = 1,2,3,4,
\end{aligned} \tag{3.18c}
$$

and

$$
\begin{aligned}
&\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}, \quad \boldsymbol{\beta} = \begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix}, \quad \boldsymbol{\phi} = \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix}, \quad \boldsymbol{\psi} = \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}, \\
&\mathbf{a}_k = \begin{Bmatrix} \mathbf{a}_u \\ \mathbf{a}_\beta \end{Bmatrix}_k, \quad \mathbf{b}_k = \begin{Bmatrix} \mathbf{b}_u \\ \mathbf{b}_\beta \end{Bmatrix}_k, \qquad k = 1,2,3,4.
\end{aligned} \tag{3.18d}
$$

In order to establish an eigenrelation like the Stroh formalism for two-dimensional problems, we re-cast $(3.9\text{b})_1$ and (3.13a) with the assist of $(3.9\text{b})_{2,3}$ into

$$\begin{bmatrix} \mathbf{Q} & -\frac{1}{2}\mathbf{I}_{43} \\ \mathbf{R}^{\mathrm{T}} & -\mathbf{I} + \frac{1}{2}\mathbf{I}_{33} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \mathbf{b} \end{Bmatrix} = \mu \begin{bmatrix} -\mathbf{R} & -\mathbf{I} + \frac{1}{2}\mathbf{I}_{44} \\ -\mathbf{T} & -\frac{1}{2}\mathbf{I}_{34} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \mathbf{b} \end{Bmatrix}, \tag{3.19a}$$

where

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_A & \mathbf{Q}_B \\ \mathbf{Q}_B & \mathbf{Q}_D \end{bmatrix}, \qquad \mathbf{R} = \begin{bmatrix} \mathbf{R}_A & \mathbf{R}_B \\ \mathbf{R}_B & \mathbf{R}_D \end{bmatrix}, \qquad \mathbf{T} = \begin{bmatrix} \mathbf{T}_A & \mathbf{T}_B \\ \mathbf{T}_B & \mathbf{T}_D \end{bmatrix}, \tag{3.19b}$$

$$\mathbf{a} = \begin{Bmatrix} \mathbf{a}_u \\ \mathbf{a}_\beta \end{Bmatrix}, \qquad \mathbf{b} = \begin{Bmatrix} \mathbf{b}_u \\ \mathbf{b}_\beta \end{Bmatrix}. \tag{3.19c}$$

In the above, $\mathbf{I}$ denotes the identity matrix, and $\mathbf{I}_{mn}$ stands for a matrix with all zero components except the mn component, for example,

$$\mathbf{I}_{34} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{I}_{44} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ldots, \text{etc.} \tag{3.20}$$

In order to see more clearly the eigenrelation shown in (3.19a), we now write down the expressions of $\mathbf{Q}_A, \mathbf{Q}_B, \ldots, \mathbf{T}_D$ defined in (3.7b) as

$$\begin{aligned} \mathbf{Q}_A &= \begin{bmatrix} A_{11} & A_{16} \\ A_{16} & A_{66} \end{bmatrix}, & \mathbf{Q}_B &= \begin{bmatrix} B_{11} & B_{16} \\ B_{16} & B_{66} \end{bmatrix}, & \mathbf{Q}_D &= \begin{bmatrix} D_{11} & D_{16} \\ D_{16} & D_{66} \end{bmatrix}, \\ \mathbf{R}_A &= \begin{bmatrix} A_{16} & A_{12} \\ A_{66} & A_{26} \end{bmatrix}, & \mathbf{R}_B &= \begin{bmatrix} B_{16} & B_{12} \\ B_{66} & B_{26} \end{bmatrix}, & \mathbf{R}_D &= \begin{bmatrix} D_{16} & D_{12} \\ D_{66} & D_{26} \end{bmatrix}, \\ \mathbf{T}_A &= \begin{bmatrix} A_{66} & A_{26} \\ A_{26} & A_{22} \end{bmatrix}, & \mathbf{T}_B &= \begin{bmatrix} B_{66} & B_{26} \\ B_{26} & B_{22} \end{bmatrix}, & \mathbf{T}_D &= \begin{bmatrix} D_{66} & D_{26} \\ D_{26} & D_{22} \end{bmatrix}. \end{aligned} \tag{3.21}$$

By expanding (3.19a) with the assist of (3.21), we observe that the 2nd and 5th equations of (3.19a) will ensure the equality $b_1^u = -\mu b_2^u$, which is also the consequence of the symmetry of in-plane forces, i.e., $N_{12} = N_{21}$ by $(3.9a)_1$. Moreover, it is observed that the 4th and 7th equations of (3.19a) are identical, which has also been noticed by Cheng and Reddy (2002). Due to the equivalence of the 4th and 7th equations, only seven independent equations remain in (3.19a). The extra independent equation may come from the equality of (3.8), which is a result of thin plate Kirchhoff assumption because the slopes β_x and β_y are not independent in the classical lamination theory both of them are related to the deflection w. According to the suggestion of Cheng and Reddy (2002), the complete eigenrelation is given by adding (3.8) with two arbitrarily different multipliers respectively to the 4th and 7th equations of (3.19a). To have a definite expression, we select these two multipliers to be −1/2 and 1/2, and the final complete eigenrelation can then be expressed as

$$\mathbf{N}_d\boldsymbol{\xi} = \mu\boldsymbol{\xi}, \tag{3.22a}$$

where

$$\mathbf{N}_d = \left(\mathbf{L}_2 + \tfrac{1}{2}\mathbf{J}_2\right)^{-1}\left(\mathbf{L}_1 + \tfrac{1}{2}\mathbf{J}_1\right), \tag{3.22b}$$

and

$$\begin{aligned} \mathbf{L}_1 &= \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{R}^T & -\mathbf{I} \end{bmatrix}, \qquad \mathbf{L}_2 = -\begin{bmatrix} \mathbf{R} & \mathbf{I} \\ \mathbf{T} & \mathbf{0} \end{bmatrix}, \qquad \boldsymbol{\xi} = \begin{Bmatrix} \mathbf{a} \\ \mathbf{b} \end{Bmatrix}, \\ \mathbf{J}_1 &= \begin{bmatrix} -\mathbf{I}_{44} & -\mathbf{I}_{43} \\ \mathbf{I}_{34} & \mathbf{I}_{33} \end{bmatrix}, \qquad \mathbf{J}_2 = \begin{bmatrix} -\mathbf{I}_{43} & \mathbf{I}_{44} \\ \mathbf{I}_{33} & -\mathbf{I}_{34} \end{bmatrix}. \end{aligned} \tag{3.22c}$$

4. Mixed formalism

In (2.4), the constitutive laws are written by expressing the resultant forces/moments in terms of mid-plane strains/curvatures, which is similar to the use of elastic constants C_{ijkl} for the elastic solids. In applications, sometimes it is convenient by using the compliances S_{ijkl}, i.e., expressing the mid-plane strains/curvatures in terms of resultant forces/moments. In this section, mixed expression will be used with mid-plane strains and moments as basic functions. The mixed formulation has been adopted by various researchers for various applications, e.g., Jones (1974), Reissner (1980) and Zienkiewicz and Taylor (1989). In order to get a clear relation about these expressions, we re-write (2.4) in the matrix form as

$$\begin{Bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{D} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon}_0 \\ \boldsymbol{\kappa} \end{Bmatrix}, \tag{4.1}$$

which may lead to the following mixed expression

$$\begin{Bmatrix} \boldsymbol{N} \\ \boldsymbol{\kappa} \end{Bmatrix} = \begin{bmatrix} \widetilde{\boldsymbol{A}} & \widetilde{\boldsymbol{B}} \\ -\widetilde{\boldsymbol{B}}^{\mathrm{T}} & \widetilde{\boldsymbol{D}} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon}_0 \\ \boldsymbol{M} \end{Bmatrix}, \tag{4.2a}$$

where

$$\widetilde{\boldsymbol{A}} = \boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{B}, \qquad \widetilde{\boldsymbol{B}} = \boldsymbol{B}\boldsymbol{D}^{-1}, \qquad \widetilde{\boldsymbol{D}} = \boldsymbol{D}^{-1}. \tag{4.2b}$$

The inversion of (4.2a), which will also be used in the following derivation, is now written as

$$\begin{Bmatrix} \boldsymbol{\varepsilon}_0 \\ \boldsymbol{M} \end{Bmatrix} = \begin{bmatrix} \widetilde{\boldsymbol{A}} & \widetilde{\boldsymbol{B}} \\ -\widetilde{\boldsymbol{B}}^{\mathrm{T}} & \widetilde{\boldsymbol{D}} \end{bmatrix}^{-1} \begin{Bmatrix} \boldsymbol{N} \\ \boldsymbol{\kappa} \end{Bmatrix} = \begin{bmatrix} \widetilde{\boldsymbol{A}}^* & \widetilde{\boldsymbol{B}}^* \\ -\widetilde{\boldsymbol{B}}^{*\mathrm{T}} & \widetilde{\boldsymbol{D}}^* \end{bmatrix} \begin{Bmatrix} \boldsymbol{N} \\ \boldsymbol{\kappa} \end{Bmatrix}, \tag{4.3a}$$

where

$$\widetilde{\boldsymbol{A}}^* = \boldsymbol{A}^{-1}, \qquad \widetilde{\boldsymbol{B}}^* = -\boldsymbol{A}^{-1}\boldsymbol{B}, \qquad \widetilde{\boldsymbol{D}}^* = \boldsymbol{D} - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}. \tag{4.3b}$$

Because $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{D}$ defined in (2.5) are symmetric matrices, $\widetilde{\boldsymbol{A}}$, $\widetilde{\boldsymbol{D}}$, $\widetilde{\boldsymbol{A}}^*$ and $\widetilde{\boldsymbol{D}}^*$ defined in (4.2b) and (4.3b) will also be symmetric, whereas $\widetilde{\boldsymbol{B}}$ and $\widetilde{\boldsymbol{B}}^*$ may not be symmetric.

Note that in the above matrix expressions, the symbols $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{N}$ have different representations from the eigenvector matrices $\mathbf{A}_d$, $\mathbf{B}_d$ defined in (3.18c) and the fundamental matrix $\mathbf{N}_d$ defined in (3.22). The former is the traditional notation used in the community of mechanics of composite materials, while the latter is the notation generally used in the community of anisotropic elasticity. To let the readers from both communities see clearly what we express in this paper, we just use the subscripts and italic fonts to distinguish these symbols.

Similar to the displacement formalism, we re-write the mixed constitutive laws (4.2) in terms of tensor notation as

$$N_{ij} = \widetilde{A}_{ijkl}\varepsilon_{kl} + \widetilde{B}_{ijkl}M_{kl}, \quad \kappa_{ij} = -\widetilde{B}_{klij}\varepsilon_{kl} + \widetilde{D}_{ijkl}M_{kl}, \qquad i,j,k,l = 1,2. \tag{4.4}$$

Note that due to the two times difference for the tensor notation and contracted notation of shear strain and twist curvature denoted in (3.2), the following rule should be followed

$$\begin{aligned}
&\widetilde{A}_{pqrs} \leftrightarrow \widetilde{A}_{ij} \quad \text{for all } i \text{ and } j, \quad i,j = 1,2,6, \\
&\widetilde{B}_{pqrs} \leftrightarrow \widetilde{B}_{ij} \quad \text{if } j \neq 6, \\
&\widetilde{B}_{pqrs} \leftrightarrow \tfrac{1}{2}\widetilde{B}_{ij} \quad \text{if } j = 6, \\
&\widetilde{D}_{pqrs} \leftrightarrow \widetilde{D}_{ij} \quad \text{if } i,j \neq 6, \\
&\widetilde{D}_{pqrs} \leftrightarrow \tfrac{1}{2}\widetilde{D}_{ij} \quad \text{if either } i \text{ or } j = 6, \\
&\widetilde{D}_{pqrs} \leftrightarrow \tfrac{1}{4}\widetilde{D}_{ij} \quad \text{if both } i \text{ and } j = 6.
\end{aligned} \tag{4.5}$$

Because the basic functions we use in the mixed constitutive laws are the strains ε_{ij} and moments M_{ij}, the kinematic relations shown in $(3.1)_2$ and the equilibrium equations shown in $(3.1)_4$ are better replaced by

$$\begin{aligned} &\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \qquad M_{ij} = \tfrac{1}{2}(\psi_{i,j^*} + \psi_{j,i^*}), \\ &N_{ij,j} = 0, \qquad \kappa_{ij,j^*} = 0, \end{aligned} \tag{4.6}$$

where the superscript $*$ denotes the coordinate system (x_1^*, x_2^*) which is related to (x_1, x_2) by

$$x_1^* = -x_2, \qquad x_2^* = x_1. \tag{4.7}$$

By (4.7), we have $\partial/\partial x_1^* = -\partial/\partial x_2$ and $\partial/\partial x_2^* = \partial/\partial x_1$. Thus, the expressions in (4.6) for M_{ij} and κ_{ij} are equivalent to

$$\begin{aligned} &M_{11} = -\psi_{1,2}, \qquad M_{22} = \psi_{2,1}, \qquad M_{12} = (\psi_{1,1} - \psi_{2,2})/2, \\ &\kappa_{11,2} - \kappa_{12,1} = 0, \qquad \kappa_{12,2} - \kappa_{22,1} = 0. \end{aligned} \tag{4.8}$$

The three equations of $(4.8)_1$ show that the moments defined in the second equation of $(4.6)_1$ will automatically satisfy the equilibrium equation

$$M_{ij,ij} = 0. \tag{4.9}$$

On the other hand, the kinematic relations for the curvatures shown in the second and third equations of $(3.1)_1$ and the third equation of $(3.1)_2$ will lead to

$$\kappa_{11} = -w_{,11}, \qquad \kappa_{22} = -w_{,22}, \qquad \kappa_{12} = -w_{,12}, \tag{4.10}$$

which will then automatically satisfy the two equations of $(4.8)_2$, i.e., the second equation of $(4.6)_2$. In other words, we may call the second equation of $(4.6)_1$ as the kinematic relation for the moments, and the second equation of $(4.6)_2$ as the compatibility equation for the curvatures.

Substituting $(4.6)_1$ into (4.4), the resultant forces and curvatures may be expressed in terms of mid-plane displacements u_i and stress function ψ_i as

$$N_{ij} = \widetilde{A}_{ijkl} u_{k,l} + \widetilde{B}_{ijkl} \psi_{k,l^*}, \qquad \kappa_{ij} = -\widetilde{B}_{klij} u_{k,l} + \widetilde{D}_{ijkl} \psi_{k,l^*}. \tag{4.11}$$

With this result, the equilibrium equations and compatibility equations shown in $(4.6)_2$ may also be expressed in terms of mid-plane displacements u_i and stress function ψ_i as

$$\widetilde{A}_{ijkl} u_{k,lj} + \widetilde{B}_{ijkl} \psi_{k,l^*j} = 0, \qquad -\widetilde{B}_{klij} u_{k,lj^*} + \widetilde{D}_{ijkl} \psi_{k,l^*j^*} = 0. \tag{4.12}$$

Like the derivation for the displacement formalism, we may now let

$$u_k = a_k^u f(z), \quad \psi_k = a_k^\psi f(z), \qquad z = x_1 + \mu x_2, \ k = 1, 2. \tag{4.13}$$

Substituting (4.13) into (4.12) with the use of (4.7), we obtain

$$\begin{aligned} &\{\mathbf{Q}_{\tilde{A}} + \mu(\mathbf{R}_{\tilde{A}} + \mathbf{R}_{\tilde{A}}^{\mathrm{T}}) + \mu^2 \mathbf{T}_{\tilde{A}}\}\mathbf{a}_u + \{\mathbf{R}_{\tilde{B}} + \mu(\mathbf{T}_{\tilde{B}} - \mathbf{Q}_{\tilde{B}}) - \mu^2 \widetilde{\mathbf{R}}_{\tilde{B}}\}\mathbf{a}_\psi = \mathbf{0}, \\ &\{-\mathbf{R}_{\tilde{B}}^{\mathrm{T}} + \mu(\mathbf{Q}_{\tilde{B}}^{\mathrm{T}} - \mathbf{T}_{\tilde{B}}^{\mathrm{T}}) + \mu^2 \widetilde{\mathbf{R}}_{\tilde{B}}^{\mathrm{T}}\}\mathbf{a}_u + \{\mathbf{T}_{\tilde{D}} - \mu(\mathbf{R}_{\tilde{D}} + \mathbf{R}_{\tilde{D}}^{\mathrm{T}}) + \mu^2 \mathbf{Q}_{\tilde{D}}\}\mathbf{a}_\psi = 0, \end{aligned} \tag{4.14a}$$

where

$$\mathbf{a}_u = \begin{Bmatrix} a_1^u \\ a_2^u \end{Bmatrix}, \qquad \mathbf{a}_\psi = \begin{Bmatrix} a_1^\psi \\ a_2^\psi \end{Bmatrix}, \tag{4.14b}$$

and the definition of $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{T}$ are the same as those given in (3.7b). Concerning the unsymmetry of $\widetilde{\mathbf{B}}$, a new matrix $\widetilde{\mathbf{R}}_{\tilde{B}}$ is defined by

$$\widetilde{\mathbf{R}}_{\tilde{B}} = \widetilde{B}_{i2k1}. \tag{4.15}$$

With the understanding of the transformation rules given in (3.2) and (4.5), we may write down the expressions of $\mathbf{Q}_{\tilde{A}}, \mathbf{Q}_{\tilde{B}}, \ldots, \mathbf{T}_{\tilde{D}}$, $\widetilde{\mathbf{R}}_{\tilde{B}}$ defined in (3.7b) and (4.15) as

$$\mathbf{Q}_{\tilde{A}}=\begin{bmatrix}\widetilde{A}_{11} & \widetilde{A}_{16}\\ \widetilde{A}_{16} & \widetilde{A}_{66}\end{bmatrix},\quad \mathbf{Q}_{\tilde{B}}=\begin{bmatrix}\widetilde{B}_{11} & \frac{1}{2}\widetilde{B}_{16}\\ \widetilde{B}_{61} & \frac{1}{2}\widetilde{B}_{66}\end{bmatrix},\quad \mathbf{Q}_{\tilde{D}}=\begin{bmatrix}\widetilde{D}_{11} & \frac{1}{2}\widetilde{D}_{16}\\ \frac{1}{2}\widetilde{D}_{16} & \frac{1}{4}\widetilde{D}_{66}\end{bmatrix},$$
$$\mathbf{R}_{\tilde{A}}=\begin{bmatrix}\widetilde{A}_{16} & \widetilde{A}_{12}\\ \widetilde{A}_{66} & \widetilde{A}_{26}\end{bmatrix},\quad \mathbf{R}_{\tilde{B}}=\begin{bmatrix}\frac{1}{2}\widetilde{B}_{16} & \widetilde{B}_{12}\\ \frac{1}{2}\widetilde{B}_{66} & \widetilde{B}_{62}\end{bmatrix},\quad \mathbf{R}_{\tilde{D}}=\begin{bmatrix}\frac{1}{2}\widetilde{D}_{16} & \widetilde{D}_{12}\\ \frac{1}{4}\widetilde{D}_{66} & \frac{1}{2}\widetilde{D}_{26}\end{bmatrix},\quad \widetilde{\mathbf{R}}_{\tilde{B}}=\begin{bmatrix}\widetilde{B}_{61} & \frac{1}{2}\widetilde{B}_{66}\\ \widetilde{B}_{21} & \frac{1}{2}\widetilde{B}_{26}\end{bmatrix},\tag{4.16}$$
$$\mathbf{T}_{\tilde{A}}=\begin{bmatrix}\widetilde{A}_{66} & \widetilde{A}_{26}\\ \widetilde{A}_{26} & \widetilde{A}_{22}\end{bmatrix},\quad \mathbf{T}_{\tilde{B}}=\begin{bmatrix}\frac{1}{2}\widetilde{B}_{66} & \widetilde{B}_{62}\\ \frac{1}{2}\widetilde{B}_{26} & \widetilde{B}_{22}\end{bmatrix},\quad \mathbf{T}_{\tilde{D}}=\begin{bmatrix}\frac{1}{4}\widetilde{D}_{66} & \frac{1}{2}\widetilde{D}_{26}\\ \frac{1}{2}\widetilde{D}_{26} & \widetilde{D}_{22}\end{bmatrix}.$$

Substituting (4.13) into (4.11), we have

$$\begin{aligned}&N_{i1}=-\mu\mathbf{b}_u f'(z),\qquad N_{i2}=\mathbf{b}_u f'(z),\\ &\kappa_{i1}=\mathbf{b}_\psi f'(z),\qquad \kappa_{i2}=\mu\mathbf{b}_\psi f'(z),\end{aligned}\tag{4.17a}$$

where

$$\begin{aligned}\mathbf{b}_u&=(\mathbf{R}_{\tilde{A}}^{\mathrm{T}}+\mu\mathbf{T}_{\tilde{A}})\mathbf{a}_u+(\mathbf{T}_{\tilde{B}}-\mu\widetilde{\mathbf{R}}_{\tilde{B}})\mathbf{a}_\psi\\ &=-\frac{1}{\mu}\{(\mathbf{Q}_{\tilde{A}}+\mu\mathbf{R}_{\tilde{A}})\mathbf{a}_u+(\mathbf{R}_{\tilde{B}}-\mu\mathbf{Q}_{\tilde{B}})\mathbf{a}_\psi\},\\ \mathbf{b}_\psi&=-(\mathbf{Q}_{\tilde{A}}^{\mathrm{T}}+\mu\widetilde{\mathbf{R}}_{\tilde{B}}^{\mathrm{T}})\mathbf{a}_u+(\mathbf{R}_{\tilde{D}}-\mu\mathbf{Q}_{\tilde{D}})\mathbf{a}_\psi\\ &=-\frac{1}{\mu}\{(\mathbf{R}_{\tilde{B}}^{\mathrm{T}}+\mu\mathbf{T}_{\tilde{B}}^{\mathrm{T}})\mathbf{a}_u-(\mathbf{T}_{\tilde{D}}-\mu\mathbf{R}_{\tilde{D}}^{\mathrm{T}})\mathbf{a}_\psi\}.\end{aligned}\tag{4.17b}$$

Note that the second equalities of $(4.17\text{b})_1$ and $(4.17\text{b})_2$ come from the two equations of (4.14a). Observing the results obtained in (4.17), we introduce one stress function ϕ_i and one slope function β_i as

$$\phi_i=b_i^u f(z),\qquad \beta_i=b_i^\psi f(z).\tag{4.18}$$

With the use of these two functions, the resultant forces and curvatures can be expressed as

$$\begin{aligned}&N_{i1}=-\phi_{i,2},\qquad N_{i2}=\phi_{i,1},\\ &\kappa_{i1}=\beta_{i,1},\qquad \kappa_{i2}=\beta_{i,2}.\end{aligned}\tag{4.19}$$

By (4.18) and (4.19), the symmetry requirement $N_{12}=N_{21}$ and $\kappa_{12}=\kappa_{21}$ may lead to

$$-\mu b_2^u=b_1^u,\qquad b_2^\psi=\mu b_1^\psi.\tag{4.20}$$

Similar to the displacement formalism, by superimposing all the associated solutions shown in (4.13) and (4.18), the general solutions to the basic equations (4.4) and (4.6) can be written in a compact matrix form as

$$\mathbf{u}_m=2\mathrm{Re}\{\mathbf{A}_m\mathbf{f}(z)\},\qquad \boldsymbol{\phi}_m=2\mathrm{Re}\{\mathbf{B}_m\mathbf{f}(z)\},\tag{4.21a}$$

where

$$\mathbf{u}_m=\begin{Bmatrix}\mathbf{u}\\ \boldsymbol{\psi}\end{Bmatrix},\qquad \boldsymbol{\phi}_m=\begin{Bmatrix}\boldsymbol{\phi}\\ \boldsymbol{\beta}\end{Bmatrix},\tag{4.21b}$$

$$\mathbf{A}_m=\begin{bmatrix}\tilde{\mathbf{a}}_1 & \tilde{\mathbf{a}}_2 & \tilde{\mathbf{a}}_3 & \tilde{\mathbf{a}}_4\end{bmatrix},\qquad \mathbf{B}_m=\begin{bmatrix}\tilde{\mathbf{b}}_1 & \tilde{\mathbf{b}}_2 & \tilde{\mathbf{b}}_3 & \tilde{\mathbf{b}}_4\end{bmatrix},\tag{4.21c}$$

and

$$\tilde{\mathbf{a}}_k = \begin{Bmatrix} \mathbf{a}_u \\ \mathbf{a}_\psi \end{Bmatrix}_k, \quad \tilde{\mathbf{b}}_k = \begin{Bmatrix} \mathbf{b}_u \\ \mathbf{b}_\psi \end{Bmatrix}_k, \qquad k = 1,2,3,4. \tag{4.21d}$$

Like the displacement formalism, (4.17b) can be re-organized into the following eigenrelation

$$\begin{bmatrix} \widetilde{\mathbf{Q}} & \mathbf{0} \\ \widetilde{\mathbf{R}}^{\mathrm{T}} & -\mathbf{I} \end{bmatrix} \begin{Bmatrix} \tilde{\mathbf{a}} \\ \tilde{\mathbf{b}} \end{Bmatrix} = \mu \begin{bmatrix} -\widetilde{\mathbf{R}} & -\mathbf{I} \\ -\widetilde{\mathbf{T}} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \tilde{\mathbf{a}} \\ \tilde{\mathbf{b}} \end{Bmatrix}, \tag{4.22a}$$

where

$$\widetilde{\mathbf{Q}} = \begin{bmatrix} \mathbf{Q}_{\tilde{A}} & \mathbf{R}_{\tilde{B}} \\ \mathbf{R}_{\tilde{B}}^{\mathrm{T}} & -\mathbf{T}_{\tilde{D}} \end{bmatrix}, \qquad \widetilde{\mathbf{R}} = \begin{bmatrix} \mathbf{R}_{\tilde{A}} & -\mathbf{Q}_{\tilde{B}} \\ \mathbf{T}_{\tilde{B}}^{\mathrm{T}} & \mathbf{R}_{\tilde{D}}^{\mathrm{T}} \end{bmatrix}, \qquad \widetilde{\mathbf{T}} = \begin{bmatrix} \mathbf{T}_{\tilde{A}} & -\widetilde{\mathbf{R}}_{\tilde{B}} \\ -\widetilde{\mathbf{R}}_{\tilde{B}}^{\mathrm{T}} & -\mathbf{Q}_{\tilde{D}} \end{bmatrix}. \tag{4.22b}$$

Better than the eigenrelation of the displacement formalism shown in (3.19a), Eq. (4.22a) has exactly the same form as that of Stroh formalism for two-dimensional problems. Therefore, all the relations originated from the eigenrelation for the two-dimensional problems can automatically be copied to the present eigenrelation of mixed formalism. By using the inverse relation

$$\begin{bmatrix} -\widetilde{\mathbf{R}} & -\mathbf{I} \\ -\widetilde{\mathbf{T}} & \mathbf{0} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{0} & -\widetilde{\mathbf{T}}^{-1} \\ -\mathbf{I} & \widetilde{\mathbf{R}}\widetilde{\mathbf{T}}^{-1} \end{bmatrix}, \tag{4.23}$$

the eigenrelation (4.22a) can then be written into the following standard eigenrelation as

$$\mathbf{N}_m \tilde{\boldsymbol{\xi}} = \mu \tilde{\boldsymbol{\xi}}, \tag{4.24a}$$

where

$$\mathbf{N}_m = \begin{bmatrix} \widetilde{\mathbf{N}}_1 & \widetilde{\mathbf{N}}_2 \\ \widetilde{\mathbf{N}}_3 & \widetilde{\mathbf{N}}_1^{\mathrm{T}} \end{bmatrix}, \qquad \tilde{\boldsymbol{\xi}} = \begin{Bmatrix} \tilde{\mathbf{a}} \\ \tilde{\mathbf{b}} \end{Bmatrix}, \tag{4.24b}$$

and

$$\widetilde{\mathbf{N}}_1 = -\widetilde{\mathbf{T}}^{-1}\widetilde{\mathbf{R}}^{\mathrm{T}}, \qquad \widetilde{\mathbf{N}}_2 = \widetilde{\mathbf{T}}^{-1} = \widetilde{\mathbf{N}}_2^{\mathrm{T}}, \qquad \widetilde{\mathbf{N}}_3 = \widetilde{\mathbf{R}}\widetilde{\mathbf{T}}^{-1}\widetilde{\mathbf{R}}^{\mathrm{T}} - \widetilde{\mathbf{Q}} = \widetilde{\mathbf{N}}_3^{\mathrm{T}}. \tag{4.24c}$$

5. Explicit expressions for the material eigenvectors and fundamental matrix

As shown in the general solution (3.18) or (4.21) and the eigenrelation (3.22) or (4.24), the material eigenvector matrices ($\mathbf{A}_d$, $\mathbf{B}_d$ for displacement formalism and $\mathbf{A}_m$, $\mathbf{B}_m$ for the mixed formalism) and the fundamental matrix ($\mathbf{N}_d$ for displacement formalism and $\mathbf{N}_m$ for the mixed formalism) play important roles in the Stroh-like formalism for the coupled stretching–bending analysis. Like the Stroh formalism for two-dimensional problems, it would be of much benefit if we can get the explicit expressions of the material eigenvectors and the fundamental matrix. If one is familiar with the Stroh formalism, one should observe that the explicit expressions of material eigenvectors for two-dimensional problems are obtained from the benefit of the stress-based Lekhnitskii formalism (Ting, 1996). Whereas, the explicit expressions of the fundamental matrices are obtained from definitions same as those given in (4.24a) for the mixed formalism not those given in (3.22b,c) for the displacement formalism (Ting, 1996). That is why until now no explicit expressions for the material eigenvectors and fundamental matrices have been obtained in the literature even the displacement formalism has been presented several years ago.

To show the benefits of the mixed formalism, the explicit expressions for the material eigenvectors and fundamental matrices will now be obtained based upon the mixed formalism presented in Section 4.

5.1. Material eigenvectors

To find the explicit expressions of material eigenvectors, we first consider the constitutive laws shown in (4.3). By using $(4.6)_1$, (4.13), (4.18), (4.19) and the relation obtained in (4.20), we get

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \\ M_{11} \\ M_{22} \\ M_{12} \end{Bmatrix} = \begin{Bmatrix} a_1^u \\ \mu a_2^u \\ \mu a_1^u + a_2^u \\ -\mu a_1^\psi \\ a_2^\psi \\ (a_1^\psi - \mu a_2^\psi)/2 \end{Bmatrix} f'(z), \qquad \begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \\ \kappa_{11} \\ \kappa_{22} \\ 2\kappa_{12} \end{Bmatrix} = \begin{Bmatrix} \mu^2 b_2^u \\ b_2^u \\ -\mu b_2^u \\ b_1^\psi \\ \mu^2 b_1^\psi \\ 2\mu b_1^\psi \end{Bmatrix} f'(z). \tag{5.1}$$

Substituting (5.1) into (4.3), and expanding its results into six equations, we obtain

$$\begin{aligned} &a_1^u = p_1 b_2^u + q_1 b_1^\psi, \quad \mu a_2^u = p_2 b_2^u + q_2 b_1^\psi, \quad \mu a_1^u + a_2^u = p_6 b_2^u + q_6 b_1^\psi, \\ &-\mu a_1^\psi = -h_1 b_2^u + g_1 b_1^\psi, \quad a_2^\psi = -h_2 b_2^u + g_2 b_1^\psi, \quad (a_1^\psi - \mu a_2^\psi)/2 = -h_6 b_2^u + g_6 b_1^\psi, \end{aligned} \tag{5.2a}$$

where

$$\begin{aligned} &p_j = \mu^2 \widetilde{A}_{j1}^* + \widetilde{A}_{j2}^* - \mu \widetilde{A}_{j6}^*, \qquad q_j = \widetilde{B}_{j1}^* + \mu^2 \widetilde{B}_{j2}^* + 2\mu \widetilde{B}_{j6}^*, \\ &h_j = \mu^2 \widetilde{B}_{1j}^* + \widetilde{B}_{2j}^* - \mu \widetilde{B}_{6j}^*, \qquad g_j = \widetilde{D}_{j1}^* + \mu^2 \widetilde{D}_{j2}^* + 2\mu \widetilde{D}_{j6}^*. \end{aligned} \tag{5.2b}$$

Since both of the three equations of $(5.2a)_1$ and $(5.2a)_2$ are not independent each other, by standard elimination procedure with proper multiplication, addition and subtraction we may obtain

$$l_1 b_2^u + l_2 b_1^\psi = 0, \qquad l_3 b_2^u - l_4 b_1^\psi = 0, \tag{5.3a}$$

where

$$\begin{aligned} &l_1 = \mu p_1 + \frac{p_2}{\mu} - p_6, \qquad l_2 = \mu q_1 + \frac{q_2}{\mu} - q_6, \\ &l_3 = \frac{h_1}{2\mu} + \frac{\mu h_2}{2} + h_6, \qquad l_4 = \frac{g_1}{2\mu} + \frac{\mu g_2}{2} + g_6. \end{aligned} \tag{5.3b}$$

Eqs. (5.3) show that nontrivial solutions of b_2^u and b_1^ψ exist only when

$$l_1(\mu) l_4(\mu) + l_2(\mu) l_3(\mu) = 0. \tag{5.4}$$

By viewing (5.2b) and (5.3b), we know that (5.4) is an 8th order polynomial which should lead to the same eigenvalues as those obtained from the eigenvalue relation (4.24). Furthermore, after obtaining the eigenvalues from (5.4), Eq. (5.3) may give us

$$b_1^\psi = \lambda b_2^u, \qquad \lambda = -\frac{l_1}{l_2} = \frac{l_3}{l_4} \quad \text{if } l_2 \text{ and/or } l_4 \neq 0; \tag{5.5a}$$

or,

$$b_2^u = \lambda^{-1} b_1^\psi, \qquad \lambda^{-1} = -\frac{l_2}{l_1} = \frac{l_4}{l_3} \quad \text{if } l_1 \text{ and/or } l_3 \neq 0. \tag{5.5b}$$

With the results of (4.20), (5.5) and (5.2), the explicit expressions for the eigenvectors $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ can then be written as

$$\tilde{\mathbf{a}} = \begin{Bmatrix} \mathbf{a}_u \\ \mathbf{a}_\psi \end{Bmatrix} = \begin{Bmatrix} p_1 + \lambda q_1 \\ (p_2 + \lambda q_2)/\mu \\ (h_1 - \lambda g_1)/\mu \\ -h_2 + \lambda g_2 \end{Bmatrix}, \quad \tilde{\mathbf{b}} = \begin{Bmatrix} \mathbf{b}_u \\ \mathbf{b}_\psi \end{Bmatrix} = \begin{Bmatrix} -\mu \\ 1 \\ \lambda \\ \lambda\mu \end{Bmatrix} \quad \text{if } l_2 \text{ and/or } l_4 \neq 0; \tag{5.6a}$$

or,

$$\tilde{\mathbf{a}} = \begin{Bmatrix} \mathbf{a}_u \\ \mathbf{a}_\psi \end{Bmatrix} = \begin{Bmatrix} \lambda^{-1} p_1 + q_1 \\ (\lambda^{-1} p_2 + q_2)/\mu \\ (\lambda^{-1} h_1 - g_1)/\mu \\ -\lambda^{-1} h_2 + g_2 \end{Bmatrix}, \quad \tilde{\mathbf{b}} = \begin{Bmatrix} \mathbf{b}_u \\ \mathbf{b}_\psi \end{Bmatrix} = \begin{Bmatrix} -\lambda^{-1}\mu \\ \lambda^{-1} \\ 1 \\ \mu \end{Bmatrix} \quad \text{if } l_1 \text{ and/or } l_3 \neq 0. \tag{5.6b}$$

5.2. Fundamental matrix

From the definition given in (4.24), we know that the fundamental matrix $\mathbf{N}_m$ of the mixed formalism is a 8×8 matrix which is related to the extensional, bending and coupling stiffness matrices. Although it looks complicated, it is not difficult to get the explicit expression because the definition of the fundamental matrix given in (4.24) has been purposely arranged to be the same as that of the two-dimensional problems. With this understanding, by following the steps described in Ting's book (1996) for Stroh formalism we can find the explicit expressions of $\mathbf{N}_m$. This benefit for the mixed formalism cannot be applied to the displacement formalism because the definition given in (3.22) is not perfectly matched with the Stroh formalism for two-dimensional problems. In Section 7, we will try to find the relation between $\mathbf{N}_m$ and $\mathbf{N}_d$ through the proper comparison of these two formalisms. With that relation, the explicit expression of $\mathbf{N}_d$ can be found easily via $\mathbf{N}_m$.

Followings are the explicit expressions of $\widetilde{\mathbf{N}}_i$, $i = 1, 2, 3$, which are the sub-matrices of $\mathbf{N}_m$. Detailed derivation may be found in (Hsieh and Hwu, 2002b).

$$\widetilde{\mathbf{N}}_1 = \frac{1}{\widetilde{\Delta}} \begin{bmatrix} X_{11} & X_{12} & 0 & X_{14} \\ X_{21} & 0 & 0 & X_{24} \\ X_{31} & 0 & 0 & X_{34} \\ X_{41} & 0 & X_{43} & X_{44} \end{bmatrix}, \quad \widetilde{\mathbf{N}}_2 = \frac{1}{\widetilde{\Delta}} \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{12} & Y_{22} & Y_{23} & Y_{24} \\ Y_{13} & Y_{23} & Y_{33} & Y_{34} \\ Y_{14} & Y_{24} & Y_{34} & Y_{44} \end{bmatrix},$$

$$\widetilde{\mathbf{N}}_3 = \frac{1}{\widetilde{\Delta}} \begin{bmatrix} -\widetilde{D}_{22}^* & 0 & 0 & \widetilde{B}_{12}^* \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \widetilde{B}_{12}^* & 0 & 0 & \widetilde{A}_{11}^* \end{bmatrix}, \tag{5.7a}$$

where

$$
\begin{aligned}
&\widetilde{\Delta} = \widetilde{B}^*_{12}\widetilde{B}^*_{12} + \widetilde{A}^*_{11}\widetilde{D}^*_{22},\\
&X_{11} = \widetilde{A}^*_{16}\widetilde{D}^*_{22} + \widetilde{B}^*_{12}\widetilde{B}^*_{62}, \quad X_{12} = -(\widetilde{A}^*_{11}\widetilde{D}^*_{22} + \widetilde{B}^*_{12}\widetilde{B}^*_{12}), \quad X_{14} = \widetilde{A}^*_{11}\widetilde{B}^*_{62} - \widetilde{A}^*_{16}\widetilde{B}^*_{12},\\
&X_{21} = \widetilde{A}^*_{12}\widetilde{D}^*_{22} + \widetilde{B}^*_{12}\widetilde{B}^*_{22}, \quad X_{24} = \widetilde{A}^*_{11}\widetilde{B}^*_{22} - \widetilde{A}^*_{12}\widetilde{B}^*_{12},\\
&X_{31} = \widetilde{B}^*_{11}\widetilde{D}^*_{22} - \widetilde{B}^*_{12}\widetilde{D}^*_{12}, \quad X_{34} = -(\widetilde{A}^*_{11}\widetilde{D}^*_{12} + \widetilde{B}^*_{11}\widetilde{B}^*_{12}),\\
&X_{41} = 2(\widetilde{B}^*_{16}\widetilde{D}^*_{22} - \widetilde{B}^*_{12}\widetilde{D}^*_{26}), \quad X_{43} = \widetilde{A}^*_{11}\widetilde{D}^*_{22} + \widetilde{B}^*_{12}\widetilde{B}^*_{12}, \quad X_{44} = -2(\widetilde{A}^*_{11}\widetilde{D}^*_{26} + \widetilde{B}^*_{16}\widetilde{B}^*_{12})\\
&Y_{11} = \widetilde{A}^*_{11}\widetilde{B}^{*2}_{62} + \widetilde{A}^*_{66}\widetilde{B}^{*2}_{12} + \widetilde{A}^*_{11}\widetilde{A}^*_{66}\widetilde{D}^*_{22} - \widetilde{A}^{*2}_{16}\widetilde{D}^*_{22} - 2\widetilde{A}^*_{16}\widetilde{B}^*_{12}\widetilde{B}^*_{62},\\
&Y_{12} = \widetilde{A}^*_{11}\widetilde{B}^*_{22}\widetilde{B}^*_{62} + \widetilde{A}^*_{26}\widetilde{B}^{*2}_{12} + \widetilde{A}^*_{11}\widetilde{A}^*_{26}\widetilde{D}^*_{22} - \widetilde{A}^*_{12}\widetilde{A}^*_{16}\widetilde{D}^*_{22} - \widetilde{A}^*_{12}\widetilde{B}^*_{12}\widetilde{B}^*_{62} - \widetilde{A}^*_{16}\widetilde{B}^*_{12}\widetilde{B}^*_{22},\\
&Y_{13} = \widetilde{A}^*_{16}\widetilde{B}^*_{12}\widetilde{D}^*_{12} + \widetilde{B}^{*2}_{12}\widetilde{B}^*_{61} + \widetilde{A}^*_{11}\widetilde{B}^*_{61}\widetilde{D}^*_{22} - \widetilde{A}^*_{16}\widetilde{B}^*_{11}\widetilde{D}^*_{22} - \widetilde{A}^*_{11}\widetilde{B}^*_{62}\widetilde{D}^*_{12} - \widetilde{B}^*_{11}\widetilde{B}^*_{12}\widetilde{B}^*_{62},\\
&Y_{14} = 2(\widetilde{A}^*_{16}\widetilde{B}^*_{12}\widetilde{D}^*_{26} + \widetilde{B}^{*2}_{12}\widetilde{B}^*_{66} + \widetilde{A}^*_{11}\widetilde{B}^*_{66}\widetilde{D}^*_{22} - \widetilde{A}^*_{16}\widetilde{B}^*_{16}\widetilde{D}^*_{22} - \widetilde{A}^*_{11}\widetilde{B}^*_{62}\widetilde{D}^*_{26} - \widetilde{B}^*_{12}\widetilde{B}^*_{16}\widetilde{B}^*_{62}),\\
&Y_{22} = \widetilde{A}^*_{11}\widetilde{B}^{*2}_{22} + \widetilde{A}^*_{22}\widetilde{B}^{*2}_{12} + \widetilde{A}^*_{11}\widetilde{A}^*_{22}\widetilde{D}^*_{22} - \widetilde{A}^{*2}_{12}\widetilde{D}^*_{22} - 2\widetilde{A}^*_{12}\widetilde{B}^*_{12}\widetilde{B}^*_{22},\\
&Y_{23} = \widetilde{A}^*_{12}\widetilde{B}^*_{12}\widetilde{D}^*_{12} + \widetilde{B}^{*2}_{12}\widetilde{B}^*_{21} + \widetilde{A}^*_{11}\widetilde{B}^*_{21}\widetilde{D}^*_{22} - \widetilde{A}^*_{12}\widetilde{B}^*_{11}\widetilde{D}^*_{22} - \widetilde{A}^*_{11}\widetilde{B}^*_{22}\widetilde{D}^*_{12} - \widetilde{B}^*_{11}\widetilde{B}^*_{12}\widetilde{B}^*_{22},\\
&Y_{24} = 2(\widetilde{A}^*_{12}\widetilde{B}^*_{12}\widetilde{D}^*_{26} + \widetilde{B}^{*2}_{12}\widetilde{B}^*_{26} + \widetilde{A}^*_{11}\widetilde{B}^*_{26}\widetilde{D}^*_{22} - \widetilde{A}^*_{12}\widetilde{B}^*_{16}\widetilde{D}^*_{22} - \widetilde{A}^*_{11}\widetilde{B}^*_{22}\widetilde{D}^*_{26} - \widetilde{B}^*_{12}\widetilde{B}^*_{16}\widetilde{B}^*_{22}),\\
&Y_{33} = \widetilde{A}^*_{11}\widetilde{D}^{*2}_{12} + 2\widetilde{B}^*_{11}\widetilde{B}^*_{12}\widetilde{D}^*_{12} - \widetilde{B}^{*2}_{11}\widetilde{D}^*_{22} - \widetilde{B}^{*2}_{12}\widetilde{D}^*_{11} - \widetilde{A}^*_{11}\widetilde{D}^*_{11}\widetilde{D}^*_{22},\\
&Y_{34} = 2(\widetilde{B}^*_{12}\widetilde{B}^*_{16}\widetilde{D}^*_{12} + \widetilde{A}^*_{11}\widetilde{D}^*_{12}\widetilde{D}^*_{26} + \widetilde{B}^*_{11}\widetilde{B}^*_{12}\widetilde{D}^*_{26} - \widetilde{A}^*_{11}\widetilde{D}^*_{16}\widetilde{D}^*_{22} - \widetilde{B}^{*2}_{12}\widetilde{D}^*_{16} - \widetilde{B}^*_{11}\widetilde{B}^*_{16}\widetilde{D}^*_{22}),\\
&Y_{44} = 4(\widetilde{A}^*_{11}\widetilde{D}^{*2}_{26} + 2\widetilde{B}^*_{12}\widetilde{B}^*_{16}\widetilde{D}^*_{26} - \widetilde{B}^{*2}_{16}\widetilde{D}^*_{22} - \widetilde{B}^{*2}_{12}\widetilde{D}^*_{66} - \widetilde{A}^*_{11}\widetilde{D}^*_{22}\widetilde{D}^*_{66}),
\end{aligned}
\tag{5.7b}
$$

in which $\widetilde{\mathbf{A}}^*$, $\widetilde{\mathbf{B}}^*$ and $\widetilde{\mathbf{D}}^*$ are defined in (4.3b).

6. Reduction to symmetric laminates

In our previous derivation, displacement or mixed formalism, no symmetry condition is required on the laminates. In practical engineering applications, it is common to design a symmetric laminate whose coupling stiffnesses B_{ij} are zero. For this kind of composite structures, the in-plane problem and plate bending problem will be decoupled. Recently, without involving the coupling conditions, I developed a Stroh-like formalism for the anisotropic plate through the Stroh–Lekhnitskii's connection (Hwu, in press). Because that formalism only considers the plate bending problems, to avoid coupling effects the anisotropic plates should have one plane of elastic symmetry located at the mid-plane of the plate. Therefore, it can only be applied directly to the symmetric laminates not the general composite laminates with coupling. Because that formalism has been checked carefully with Lekhnitskii formulation (1968) and applied to solve many practical problems such as holes/cracks/inclusions (Hsieh and Hwu, 2002a), it should be suitable to use that formalism as a check of our present formalisms by just reducing our results to the symmetric laminates.

6.1. Displacement formalism

Substituting $B_{ij} = 0$ into (3.21), we obtain $\mathbf{Q}_B = \mathbf{R}_B = \mathbf{T}_B = \mathbf{0}$. With this result and the definitions of $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{T}$ given in (3.19b), the eigenrelation (3.22) can now be separated into two parts as

$$
\begin{bmatrix} -\mathbf{R}_A & -\mathbf{I} \\ -\mathbf{T}_A & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{Q}_A & \mathbf{0} \\ \mathbf{R}_A^{\mathrm{T}} & -\mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{a}_u \\ \mathbf{b}_u \end{Bmatrix} = \mu \begin{Bmatrix} \mathbf{a}_u \\ \mathbf{b}_u \end{Bmatrix}, \tag{6.1}
$$

and

$$\begin{bmatrix} -\mathbf{R}_D - \frac{1}{2}\mathbf{I}_{21} & -\mathbf{I} + \frac{1}{2}\mathbf{I}_{22} \\ -\mathbf{T}_D + \frac{1}{2}\mathbf{I}_{11} & -\frac{1}{2}\mathbf{I}_{12} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{Q}_D - \frac{1}{2}\mathbf{I}_{22} & -\frac{1}{2}\mathbf{I}_{21} \\ \mathbf{R}_D^{\mathrm{T}} + \frac{1}{2}\mathbf{I}_{12} & -\mathbf{I} + \frac{1}{2}\mathbf{I}_{11} \end{bmatrix} \begin{Bmatrix} \mathbf{a}_\beta \\ \mathbf{b}_\beta \end{Bmatrix} = \mu \begin{Bmatrix} \mathbf{a}_\beta \\ \mathbf{b}_\beta \end{Bmatrix}. \tag{6.2}$$

In the above, (6.1) corresponds to the in-plane problems, while (6.2) corresponds to the plate bending problems. By careful comparison, we see that (6.1) is identical to that for two-dimensional problems (Ting, 1996), but (6.2) *looks* different. Substituting (3.21) into (6.2) and performing 4×4 matrix inversion and multiplication carefully (this work may be manipulated with the assist of symbolic computational software such as *Mathematica*), we obtain

$$\mathbf{N}_\beta \boldsymbol{\xi}_\beta = \mu \boldsymbol{\xi}_\beta, \qquad \boldsymbol{\xi}_\beta = \begin{Bmatrix} \mathbf{a}_\beta \\ \mathbf{b}_\beta \end{Bmatrix}, \tag{6.3a}$$

where

$$\mathbf{N}_\beta = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{D_{12}}{D_{22}} & -2\frac{D_{26}}{D_{22}} & 0 & \frac{1}{D_{22}} \\ -D_{11} + \frac{D_{12}^2}{D_{22}} & -2\left(D_{16} - \frac{D_{12}D_{26}}{D_{22}}\right) & 0 & -\frac{D_{12}}{D_{22}} \\ -2\left(D_{16} - \frac{D_{12}D_{26}}{D_{22}}\right) & -4\left(D_{66} - \frac{D_{26}^2}{D_{22}}\right) & 1 & -2\frac{D_{26}}{D_{22}} \end{bmatrix}. \tag{6.3b}$$

In order to check the result (6.3) with that presented in my previous paper for the symmetric laminates (Hwu, in press), we need to unify the definitions of the eigenvectors. With this consideration, it can easily be proved that $\mathbf{N}_\beta$ obtained in (6.3b) is identical to that shown in (Hwu, in press). A convenient reference for the comparison is shown in Tables 1 and 2, which will be discussed in detail in the next section.

Because the eigenvalues and eigenvectors for the symmetric laminates can be separated into two parts, in-plane problem and plate bending problem, the general solutions shown in (3.18) should also be separated into these two parts. They are

$$\begin{aligned} \mathbf{u} &= 2\mathrm{Re}\{\mathbf{A}_u \mathbf{f}(z)\}, & \boldsymbol{\phi} &= 2\mathrm{Re}\{\mathbf{B}_u \mathbf{f}(z)\}, \\ \boldsymbol{\beta} &= 2\mathrm{Re}\{\mathbf{A}_\beta \mathbf{f}(z)\}, & \boldsymbol{\psi} &= 2\mathrm{Re}\{\mathbf{B}_\beta \mathbf{f}(z)\}, \end{aligned} \tag{6.4a}$$

where

$$\begin{aligned} \mathbf{A}_u &= [\,(\mathbf{a}_u)_1 \quad (\mathbf{a}_u)_2\,], & \mathbf{B}_u &= [\,(\mathbf{b}_u)_1 \quad (\mathbf{b}_u)_2\,], \\ \mathbf{A}_\beta &= [\,(\mathbf{a}_\beta)_1 \quad (\mathbf{a}_\beta)_2\,], & \mathbf{B}_\beta &= [\,(\mathbf{b}_\beta)_1 \quad (\mathbf{b}_\beta)_2\,], \end{aligned} \tag{6.4b}$$

and

$$\mathbf{f}(z) = \begin{Bmatrix} f_1(z_1) \\ f_2(z_2) \end{Bmatrix}. \tag{6.4c}$$

From the solutions shown in (6.4) for the in-plane and plate bending problems, we see that they have exactly the same form as that for the Stroh formalism of two-dimensional problems. Again, the formalism for the plate bending problem, $(6.4a)_2$ and $(6.4b)_2$, is exactly the same as that presented in (Hwu, in press). One should note that in (Hwu, in press), the definitions for the slope vectors $\boldsymbol{\beta}$ and the stress function vector

Table 1
Comparison of general solutions

	Stroh formalism for two-dimensional problem (in-plane and antiplane coupling) (Ting, 1996)	Stroh-like formalism for plate bending problem (Hwu, in press; Hsieh and Hwu, 2002a)	Displacement formalism (in-plane and plate bending coupling)	Mixed formalism (in-plane and plate bending coupling)
General solution	$\mathbf{u}=2\mathrm{Re}\{\mathbf{Af}(z)\}$ $\boldsymbol{\phi}=2\mathrm{Re}\{\mathbf{Bf}(z)\}$	$\boldsymbol{\psi}=2\mathrm{Re}\{\mathbf{Aw}'(z)\}$ $\boldsymbol{\beta}=2\mathrm{Re}\{\mathbf{Bw}'(z)\}$	$\mathbf{u}_d=2\mathrm{Re}\{\mathbf{A}_d\mathbf{f}(z)\}$ $\boldsymbol{\phi}_d=2\mathrm{Re}\{\mathbf{B}_d\mathbf{f}(z)\}$ $\mathbf{u}_d=\begin{Bmatrix}\mathbf{u}\\ \boldsymbol{\beta}\end{Bmatrix},\ \boldsymbol{\phi}_d=\begin{Bmatrix}\boldsymbol{\phi}\\ \boldsymbol{\psi}\end{Bmatrix}$	$\mathbf{u}_m=2\mathrm{Re}\{\mathbf{A}_m\mathbf{f}(z)\}$ $\boldsymbol{\phi}_m=2\mathrm{Re}\{\mathbf{B}_m\mathbf{f}(z)\}$ $\mathbf{u}_m=\begin{Bmatrix}\mathbf{u}\\ \boldsymbol{\psi}\end{Bmatrix},\ \boldsymbol{\phi}_m=\begin{Bmatrix}\boldsymbol{\phi}\\ \boldsymbol{\beta}\end{Bmatrix}$
	$\mathbf{A}=[\mathbf{a}_1\quad \mathbf{a}_2\quad \mathbf{a}_3]$ $\mathbf{B}=[\mathbf{b}_1\quad \mathbf{b}_2\quad \mathbf{b}_3]$	$\mathbf{A}=[\mathbf{a}_1\quad \mathbf{a}_2]$ $\mathbf{B}=[\mathbf{b}_1\quad \mathbf{b}_2]$	$\mathbf{A}_d=[\mathbf{a}_1\quad \mathbf{a}_2\quad \mathbf{a}_3\quad \mathbf{a}_4]$ $\mathbf{B}_d=[\mathbf{b}_1\quad \mathbf{b}_2\quad \mathbf{b}_3\quad \mathbf{b}_4]$ $\mathbf{a}=\begin{Bmatrix}\mathbf{a}_u\\ \mathbf{a}_\beta\end{Bmatrix},\ \mathbf{b}=\begin{Bmatrix}\mathbf{b}_u\\ \mathbf{b}_\beta\end{Bmatrix}$	$\mathbf{A}_m=[\tilde{\mathbf{a}}_1\quad \tilde{\mathbf{a}}_2\quad \tilde{\mathbf{a}}_3\quad \tilde{\mathbf{a}}_4]$ $\mathbf{B}_m=[\tilde{\mathbf{b}}_1\quad \tilde{\mathbf{b}}_2\quad \tilde{\mathbf{b}}_3\quad \tilde{\mathbf{b}}_4]$ $\tilde{\mathbf{a}}=\begin{Bmatrix}\mathbf{a}_u\\ \mathbf{a}_\psi\end{Bmatrix},\ \tilde{\mathbf{b}}=\begin{Bmatrix}\mathbf{b}_u\\ \mathbf{b}_\psi\end{Bmatrix}$
	$\mathbf{f}(z)=\begin{Bmatrix}f_1(z_1)\\ f_2(z_2)\\ f_3(z_3)\end{Bmatrix}$	$\mathbf{w}'(\mathbf{z})=\begin{Bmatrix}w_1'(z_1)\\ w_2'(z_2)\end{Bmatrix}$	$\mathbf{f}(z)=\begin{Bmatrix}f_1(z_1)\\ f_2(z_2)\\ f_3(z_3)\\ f_4(z_4)\end{Bmatrix}$	$\mathbf{f}(z)=\begin{Bmatrix}f_1(z_1)\\ f_2(z_2)\\ f_3(z_3)\\ f_4(z_4)\end{Bmatrix}$
	$z_k=x_1+\mu_k x_2$	$z_k=x+\mu_k y$	$z_k=x_1+\mu_k x_2$	$z_k=x_1+\mu_k x_2$
	$\sigma_{i1}=-\phi_{i,2},\ \ \sigma_{i2}=\phi_{i,1}$	$\boldsymbol{\psi}=\begin{Bmatrix}-\int M_y\,\mathrm{d}x\\ -\int M_x\,\mathrm{d}y\end{Bmatrix}$ $\boldsymbol{\beta}=\begin{Bmatrix}-\frac{\partial w}{\partial y}\\ \frac{\partial w}{\partial x}\end{Bmatrix}$	$\beta_1=-w_{,1},\ \ \beta_2=-w_{,2}$ $N_{i1}=-\phi_{i,2},\ \ N_{i2}=\phi_{i,1}$ $M_{i1}=-\psi_{i,2}-\frac{1}{2}\lambda_{i1}\psi_{k,k},$ $M_{i2}=\psi_{i,1}-\frac{1}{2}\lambda_{i2}\psi_{k,k}$ $Q_1=-\frac{1}{2}\psi_{k,k2},\ \ Q_2=\frac{1}{2}\psi_{k,k1}$ $V_1=-\psi_{2,22},\ \ V_2=\psi_{1,11}$	$\beta_1=-w_{,1},\ \ \beta_2=-w_{,2}$ $N_{i1}=-\phi_{i,2},\ \ N_{i2}=\phi_{i,1}$ $M_{i1}=-\psi_{i,2}-\frac{1}{2}\lambda_{i1}\psi_{k,k},$ $M_{i2}=\psi_{i,1}-\frac{1}{2}\lambda_{i2}\psi_{k,k}$ $Q_1=-\frac{1}{2}\psi_{k,k2},\ \ Q_2=\frac{1}{2}\psi_{k,k1}$ $V_1=-\psi_{2,22},\ \ V_2=\psi_{1,11}$
		$M_n=-\mathbf{s}^{\mathrm{T}}\boldsymbol{\psi}_{,s},\ \ M_t=-\mathbf{n}^{\mathrm{T}}\boldsymbol{\psi}_{,n}$ $M_{nt}=(\mathbf{s}^{\mathrm{T}}\boldsymbol{\psi}_{,n}+\mathbf{n}^{\mathrm{T}}\boldsymbol{\psi}_{,s})/2$ $Q_n=-(\mathbf{s}^{\mathrm{T}}\boldsymbol{\psi}_{,ns}-\mathbf{n}^{\mathrm{T}}\boldsymbol{\psi}_{,ss})/2$ $Q_t=(\mathbf{s}^{\mathrm{T}}\boldsymbol{\psi}_{,nn}-\mathbf{n}^{\mathrm{T}}\boldsymbol{\psi}_{,sn})/2$ $V_n=-\mathbf{n}^{\mathrm{T}}\boldsymbol{\psi}_{,ss},\ \ V_t=\mathbf{s}^{\mathrm{T}}\boldsymbol{\psi}_{,nn}$		

Table 2
Comparison of eigenrelations

	Stroh formalism for two-dimensional problem (in-plane and antiplane coupling) (Ting, 1996)	Stroh-like formalism for plate bending problem (Hwu, in press)	Displacement formalism (in-plane and plate bending coupling)	Mixed formalism (in-plane and plate bending coupling)
Eigen relation	$\mathbf{N}\boldsymbol{\xi} = \mu\boldsymbol{\xi}$ $\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^{\mathrm{T}} \end{bmatrix}$, $\boldsymbol{\xi} = \begin{Bmatrix} \mathbf{a} \\ \mathbf{b} \end{Bmatrix}$	$\mathbf{N}\boldsymbol{\xi} = \mu\boldsymbol{\xi}$ $\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^{\mathrm{T}} \end{bmatrix}$, $\boldsymbol{\xi} = \begin{Bmatrix} \mathbf{a} \\ \mathbf{b} \end{Bmatrix}$	$\mathbf{N}_d\boldsymbol{\xi} = \mu\boldsymbol{\xi}$ $\mathbf{N}_d = \left(\mathbf{L}_2 + \frac{1}{2}\mathbf{J}_2\right)^{-1}\left(\mathbf{L}_1 + \frac{1}{2}\mathbf{J}_1\right)$, $\boldsymbol{\xi} = \begin{Bmatrix} \mathbf{a} \\ \mathbf{b} \end{Bmatrix}$	$\mathbf{N}_m\tilde{\boldsymbol{\xi}} = \mu\tilde{\boldsymbol{\xi}}$ $\mathbf{N}_m = \begin{bmatrix} \widetilde{\mathbf{N}}_1 & \widetilde{\mathbf{N}}_2 \\ \widetilde{\mathbf{N}}_3 & \widetilde{\mathbf{N}}_1^{\mathrm{T}} \end{bmatrix}$, $\tilde{\boldsymbol{\xi}} = \begin{Bmatrix} \tilde{\mathbf{a}} \\ \tilde{\mathbf{b}} \end{Bmatrix}$
	$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}}$, $\mathbf{N}_2 = \mathbf{T}^{-1}$	$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}}$, $\mathbf{N}_2 = \mathbf{T}^{-1}$	$\mathbf{L}_1 = \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{R}^{\mathrm{T}} & -\mathbf{I} \end{bmatrix}$, $\mathbf{L}_2 = -\begin{bmatrix} \mathbf{R} & \mathbf{I} \\ \mathbf{T} & \mathbf{0} \end{bmatrix}$,	$\widetilde{\mathbf{N}}_1 = -\widetilde{\mathbf{T}}^{-1}\widetilde{\mathbf{R}}^{\mathrm{T}}$, $\widetilde{\mathbf{N}}_2 = \widetilde{\mathbf{T}}^{-1}$,
	$\mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}} - \mathbf{Q}$	$\mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}} - \mathbf{Q}$	$\mathbf{J}_1 = \begin{bmatrix} -\mathbf{I}_{44} & -\mathbf{I}_{43} \\ \mathbf{I}_{34} & \mathbf{I}_{33} \end{bmatrix}$, $\mathbf{J}_2 = \begin{bmatrix} -\mathbf{I}_{43} & \mathbf{I}_{44} \\ \mathbf{I}_{33} & -\mathbf{I}_{34} \end{bmatrix}$	$\widetilde{\mathbf{N}}_3 = \widetilde{\mathbf{R}}\widetilde{\mathbf{T}}^{-1}\widetilde{\mathbf{R}}^{\mathrm{T}} - \widetilde{\mathbf{Q}}$
	$Q_{ik} = C_{i1k1}$, $R_{ik} = C_{i1k2}$, $T_{ik} = C_{i2k2}$ C_{ijkl}: elastic constants	$\mathbf{Q} = \begin{bmatrix} D_{22}^* & -\frac{1}{2}D_{26}^* \\ -\frac{1}{2}D_{26}^* & \frac{1}{4}D_{66}^* \end{bmatrix}$, $\mathbf{R} = \begin{bmatrix} -\frac{1}{2}D_{26}^* & D_{12}^* \\ \frac{1}{4}D_{66}^* & -\frac{1}{2}D_{16}^* \end{bmatrix}$, $\mathbf{T} = \begin{bmatrix} \frac{1}{4}D_{66}^* & -\frac{1}{2}D_{16}^* \\ -\frac{1}{2}D_{16}^* & D_{11}^* \end{bmatrix}$	$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_A & \mathbf{Q}_B \\ \mathbf{Q}_B & \mathbf{Q}_D \end{bmatrix}$, $\mathbf{R} = \begin{bmatrix} \mathbf{R}_A & \mathbf{R}_B \\ \mathbf{R}_B & \mathbf{R}_D \end{bmatrix}$, $\mathbf{T} = \begin{bmatrix} \mathbf{T}_A & \mathbf{T}_B \\ \mathbf{T}_B & \mathbf{T}_D \end{bmatrix}$	$\widetilde{\mathbf{Q}} = \begin{bmatrix} \mathbf{Q}_{\tilde{A}} & \mathbf{R}_{\tilde{B}} \\ \mathbf{R}_{\tilde{B}}^{\mathrm{T}} & -\mathbf{T}_{\tilde{D}} \end{bmatrix}$, $\widetilde{\mathbf{R}} = \begin{bmatrix} \mathbf{R}_{\tilde{A}} & -\mathbf{Q}_{\tilde{B}} \\ \mathbf{T}_{\tilde{B}}^{\mathrm{T}} & \mathbf{R}_{\tilde{D}}^{\mathrm{T}} \end{bmatrix}$, $\widetilde{\mathbf{T}} = \begin{bmatrix} \mathbf{T}_{\tilde{A}} & -\widetilde{\mathbf{R}}_{\tilde{B}} \\ -\widetilde{\mathbf{R}}_{\tilde{B}}^{\mathrm{T}} & -\mathbf{Q}_{\tilde{D}} \end{bmatrix}$
			$\mathbf{Q}_A = \begin{bmatrix} A_{11} & A_{16} \\ A_{16} & A_{66} \end{bmatrix}$, $\mathbf{Q}_B = \begin{bmatrix} B_{11} & B_{16} \\ B_{16} & B_{66} \end{bmatrix}$, $\mathbf{Q}_D = \begin{bmatrix} D_{11} & D_{16} \\ D_{16} & D_{66} \end{bmatrix}$	$\mathbf{Q}_{\tilde{A}} = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{16} \\ \widetilde{A}_{16} & \widetilde{A}_{66} \end{bmatrix}$, $\mathbf{Q}_{\tilde{B}} = \begin{bmatrix} \widetilde{B}_{11} & \frac{1}{2}\widetilde{B}_{16} \\ \widetilde{B}_{61} & \frac{1}{2}\widetilde{B}_{66} \end{bmatrix}$, $\mathbf{Q}_{\tilde{D}} = \begin{bmatrix} \widetilde{D}_{11} & \frac{1}{2}\widetilde{D}_{16} \\ \frac{1}{2}\widetilde{D}_{16} & \frac{1}{4}\widetilde{D}_{66} \end{bmatrix}$
			$\mathbf{R}_A = \begin{bmatrix} A_{16} & A_{12} \\ A_{66} & A_{26} \end{bmatrix}$, $\mathbf{R}_B = \begin{bmatrix} B_{16} & B_{12} \\ B_{66} & B_{26} \end{bmatrix}$, $\mathbf{R}_D = \begin{bmatrix} D_{16} & D_{12} \\ D_{66} & D_{26} \end{bmatrix}$	$\mathbf{R}_{\tilde{A}} = \begin{bmatrix} \widetilde{A}_{16} & \widetilde{A}_{12} \\ \widetilde{A}_{66} & \widetilde{A}_{26} \end{bmatrix}$, $\mathbf{R}_{\tilde{B}} = \begin{bmatrix} \frac{1}{2}\widetilde{B}_{16} & \widetilde{B}_{12} \\ \frac{1}{2}\widetilde{B}_{66} & \widetilde{B}_{62} \end{bmatrix}$, $\mathbf{R}_{\tilde{D}} = \begin{bmatrix} \frac{1}{2}\widetilde{D}_{16} & \widetilde{D}_{12} \\ \frac{1}{4}\widetilde{D}_{66} & \frac{1}{2}\widetilde{D}_{26} \end{bmatrix}$

	$\mathbf{T}_A = \begin{bmatrix} A_{66} & A_{26} \\ A_{26} & A_{22} \end{bmatrix}, \quad \mathbf{T}_B = \begin{bmatrix} B_{66} & B_{26} \\ B_{26} & B_{22} \end{bmatrix},$ $\mathbf{T}_D = \begin{bmatrix} D_{66} & D_{26} \\ D_{26} & D_{22} \end{bmatrix}$	
D^*_{ij}: inverse bending stiffness $\boldsymbol{D}^{-1}$	A_{ij}: extensional stiffness B_{ij}: coupling stiffness D_{ij}: bending stiffness	$\mathbf{T}_{\tilde{A}} = \begin{bmatrix} \widetilde{A}_{66} & \widetilde{A}_{26} \\ \widetilde{A}_{26} & \widetilde{A}_{22} \end{bmatrix}, \; \mathbf{T}_{\tilde{B}} = \begin{bmatrix} \frac{1}{2}\widetilde{B}_{66} & \widetilde{B}_{62} \\ \frac{1}{2}\widetilde{B}_{26} & \widetilde{B}_{22} \end{bmatrix},$ $\mathbf{T}_{\tilde{D}} = \begin{bmatrix} \frac{1}{4}\widetilde{D}_{66} & \frac{1}{2}\widetilde{D}_{26} \\ \frac{1}{2}\widetilde{D}_{26} & \widetilde{D}_{22} \end{bmatrix}$ $\widetilde{\mathbf{R}}_{\tilde{B}} = \begin{bmatrix} \widetilde{B}_{61} & \frac{1}{2}\widetilde{B}_{66} \\ \widetilde{B}_{21} & \frac{1}{2}\widetilde{B}_{26} \end{bmatrix}$ $\widetilde{\boldsymbol{A}} = \boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{B}, \; \widetilde{\boldsymbol{B}} = \boldsymbol{B}\boldsymbol{D}^{-1}, \; \widetilde{\boldsymbol{D}} = \boldsymbol{D}^{-1}$

$\boldsymbol{\psi}$ and their associated eigenvectors $\mathbf{A}_\beta$ and $\mathbf{B}_\beta$ are slight different from the present ones. The definitions given in (Hwu, in press) can also be found in Tables 1 and 2.

6.2. Mixed formalism

Substituting $B_{ij} = 0$ into (4.2), we obtain $\widetilde{\boldsymbol{A}} = \boldsymbol{A}$, $\widetilde{\boldsymbol{B}} = \mathbf{0}$, $\widetilde{\boldsymbol{D}} = \boldsymbol{D}^{-1}$, which will then lead to $\mathbf{Q}_{\tilde{B}} = \mathbf{R}_{\tilde{B}} = \widetilde{\mathbf{R}}_{\tilde{B}} = \mathbf{T}_{\tilde{B}} = \mathbf{0}$ by (4.16). With this result and the definitions of $\widetilde{\mathbf{Q}}$, $\widetilde{\mathbf{R}}$ and $\widetilde{\mathbf{T}}$ given in (4.22b), the eigenrelation (4.24) can now be separated into two parts as

$$\begin{bmatrix} -\mathbf{T}_A^{-1}\mathbf{R}_A^{\mathrm{T}} & \mathbf{T}_A^{-1} \\ \mathbf{R}_A\mathbf{T}_A^{-1}\mathbf{R}_A^{\mathrm{T}} - \mathbf{Q}_A & -\mathbf{R}_A\mathbf{T}_A^{-1} \end{bmatrix} \begin{Bmatrix} \mathbf{a}_u \\ \mathbf{b}_u \end{Bmatrix} = \mu \begin{Bmatrix} \mathbf{a}_u \\ \mathbf{b}_u \end{Bmatrix}, \tag{6.5}$$

and

$$\begin{bmatrix} \mathbf{Q}_{\tilde{D}}^{-1}\mathbf{R}_{\tilde{D}} & -\mathbf{Q}_{\tilde{D}}^{-1} \\ -\mathbf{R}_{\tilde{D}}^{\mathrm{T}}\mathbf{Q}_{\tilde{D}}^{-1}\mathbf{R}_{\tilde{D}} + \mathbf{T}_{\tilde{D}} & \mathbf{R}_{\tilde{D}}^{\mathrm{T}}\mathbf{Q}_{\tilde{D}}^{-1} \end{bmatrix} \begin{Bmatrix} \mathbf{a}_\psi \\ \mathbf{b}_\psi \end{Bmatrix} = \mu \begin{Bmatrix} \mathbf{a}_\psi \\ \mathbf{b}_\psi \end{Bmatrix}. \tag{6.6}$$

Similar to the displacement formalism, eigenrelations (6.5) and (6.6) correspond to the in-plane and plate bending problems respectively. By simple inversion and multiplication, it can easily be proved that (6.1) and (6.5) are equivalent. To prove that (6.6) is identical to (6.2), we firstly note that

$$\mathbf{a}_\psi = \mathbf{b}_\beta, \qquad \mathbf{b}_\psi = \mathbf{a}_\beta, \tag{6.7}$$

which can be observed from (3.5), (3.15), (4.13) and (4.18). With this understanding, through the use of (4.16) we can prove that (6.6) is identical to (6.2). Actually, (6.6) has exactly the same form as that presented in (Hwu, in press), and hence the proof of its equivalence is more direct and simple than (6.2).

Similar to the displacement formalism, the general solutions shown in (4.21) can also be separated into two parts, which are exactly the same as those shown in (6.4).

Because the explicit expressions for the eigenvectors have been obtained in the mixed formalism, in the following we like to check our results by using the case of symmetric laminates. Substituting $B_{ij} = 0$ into (4.3b), we have $\widetilde{\boldsymbol{A}}^* = \boldsymbol{A}^{-1}$, $\widetilde{\boldsymbol{B}}^* = \mathbf{0}$, $\widetilde{\boldsymbol{D}}^* = \boldsymbol{D}$. If we use A_{ij}^* to denote the components of $\boldsymbol{A}^{-1}$, (5.2b) gives us

$$q_j = h_j = 0, \quad p_j = \mu^2 A_{j1}^* + A_{j2}^* - \mu A_{j6}^*, \quad g_j = D_{j1} + \mu^2 D_{j2} + 2\mu D_{j6}, \tag{6.8}$$

which will then lead to, by the use of (5.3b),

$$l_2 = l_3 = 0. \tag{6.9}$$

With this result, the characteristic equation for the eigenvalues shown in (5.4) becomes

$$l_1(\mu) l_4(\mu) = 0. \tag{6.10}$$

In the above, $l_1(\mu) = 0$ will provide the eigenvalues for the in-plane problems, whereas $l_4(\mu) = 0$ will provide the eigenvalues for the plate bending problems. The explicit expressions for the eigenvectors can therefore be separated into two parts. One is from (5.6a), and the other is from (5.6b). They are

$$\mathbf{a}_u = \begin{Bmatrix} p_1 \\ p_2/\mu \end{Bmatrix}, \quad \mathbf{b}_u = \begin{Bmatrix} -\mu \\ 1 \end{Bmatrix} \quad \text{for in-plane problems,} \tag{6.11a}$$

$$\mathbf{a}_\psi = \begin{Bmatrix} -g_1/\mu \\ g_2 \end{Bmatrix}, \quad \mathbf{b}_\psi = \begin{Bmatrix} 1 \\ \mu \end{Bmatrix} \quad \text{for plate bending problems,} \tag{6.11b}$$

which are identical to those shown in Ting (1996) for the in-plane problems and in Hwu (in press) for the plate bending problems.

Substituting $\widetilde{\boldsymbol{A}}^* = \boldsymbol{A}^{-1}$, $\widetilde{\boldsymbol{B}}^* = \boldsymbol{0}$, $\widetilde{\boldsymbol{D}}^* = \boldsymbol{D}$ into (5.7) for the symmetric laminates, we can also prove that the two separate parts of the explicit expressions of the fundamental matrices corresponding to the in-plane and bending problems are exactly the same as those presented in the literature (Ting, 1996; Hwu, in press).

7. Comparison and discussion

In Ting's book (1996) and several research works, we observe that through the use of the eigenrelation many useful identities relating the material properties to the eigenmodes of stress functions and displacements can be established. With the assist of these identities, many problems that are left with unsolved linear algebraic system can be solved explicitly. Moreover, many complex variable form solutions may be transformed to real form solutions. With this understanding, in this section the comparison will be emphasized upon the resemblance of the general solutions and their associated eigenrelations, because the more alike to the Stroh formalism the more possible we can benefit from the experience of two-dimensional problems.

Table 1 shows the comparison between the general solutions presented by the Stroh formalism for two-dimensional problem (Ting, 1996), Stroh-like formalism for plate bending problem (Hwu, in press), and displacement formalism and mixed formalism presented in this paper. From this Table, we see that the Stroh-like formalism for the plate bending problem is really very alike to the Stroh formalism for two-dimensional problem. The slight difference comes from: (1) the appearance of minus sign and the order of (y, x) instead of (x, y) in the definitions of the stress function vector $\boldsymbol{\psi}$ and slope vector $\boldsymbol{\beta}$; (2) the eigenvector matrix $\mathbf{A}$ corresponds to the stress function vector instead of the slope vector, and eigenvector matrix $\mathbf{B}$ corresponds to the slope vector instead of the stress function vector. As to the displacement formalism, its solution form is exactly the same as that of Stroh formalism for two-dimensional problem. While for the mixed formalism, the in-plane part is still exactly the same as Stroh formalism for two-dimensional problem, but its plate bending part conforms to the Stroh-like formalism with $\mathbf{A}$ corresponding to the stress function and $\mathbf{B}$ corresponding to the slope. Note that in mixed formalism, the sign and the order of the definitions of $\boldsymbol{\psi}$ and $\boldsymbol{\beta}$ have been returned to the normal situation.

Purely from the observation shown in Table 1, one may conclude that displacement formalism should be the one most alike to the Stroh formalism for two-dimensional problem. However, Table 2 presenting the eigenrelation shows the opposite. From Table 2, we see that the eigenrelation of Stroh-like formalism for plate bending problem as well as that of mixed formalism have exactly the same form as that for two-dimensional problem. However, the eigenrelation for displacement formalism is different by the addition of the matrices $\mathbf{J}_1$ and $\mathbf{J}_2$. Because the matrices $\mathbf{N}_1$, $\mathbf{N}_2$ and $\mathbf{N}_3$ play an important role in the Stroh formalism for two-dimensional problems, they are usually called fundamental matrices. Therefore, although the displacement formalism has exactly the same form in the general solution, sometimes it may not be a good choice for the lamination theory due to the lack resemblance of its eigenrelation. Thus, from the viewpoint of the eigenrelation, the mixed formalism is a better choice than the displacement formalism for solving the practical lamination problems. However, because the generalized displacement vector $\mathbf{u}_m$ and the generalized stress function vector $\boldsymbol{\phi}_m$ in mixed formalism have their mix nature, it may become inconvenient when one deals with the pure stress or displacement boundary valued problems. On the other hand, if a mixed boundary valued problem (prescribed in-plane displacements and out-of-plane bending moments/effective transverse shear forces, or prescribed in-plane forces and out-of-plane deflections/slopes) is considered, mixed formalism may be a good choice.

From the above discussion, we know that both the displacement and mixed formalisms are not perfectly alike to the Stroh formalism for two-dimensional problems. One is alike in general solution, the other is alike in eigenrelation. To combine the merits from both formalisms, we may use the general solutions formed by the displacement formalism and when there is a need to count on the eigenrelation we may use the eigenrelation from the mixed formalism. From this viewpoint, we need to know the relation between the fundamental matrices $\mathbf{N}_d$ and $\mathbf{N}_m$. By (6.7), we get the following relation

$$\tilde{\boldsymbol{\xi}} = \begin{Bmatrix} \tilde{\mathbf{a}} \\ \tilde{\mathbf{b}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{a}_u \\ \mathbf{a}_\psi \\ \mathbf{b}_u \\ \mathbf{b}_\psi \end{Bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{a}_u \\ \mathbf{a}_\beta \\ \mathbf{b}_u \\ \mathbf{b}_\beta \end{Bmatrix} = \begin{bmatrix} \mathbf{I}_1 & \mathbf{I}_2 \\ \mathbf{I}_2 & \mathbf{I}_1 \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \mathbf{b} \end{Bmatrix} = \mathbf{I}_t \boldsymbol{\xi}, \tag{7.1}$$

in which $\mathbf{I}_1$, $\mathbf{I}_2$ and $\mathbf{I}_t$ are defined through the equalities. Substituting (7.1) into (4.24) and comparing its results with (3.22), we obtain

$$\mathbf{N}_d = \mathbf{I}_t \mathbf{N}_m \mathbf{I}_t, \tag{7.2}$$

or in the sub-matrix form,

$$\mathbf{N}_d = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^{\mathrm{T}} \end{bmatrix}, \tag{7.3a}$$

where

$$\begin{aligned} \mathbf{N}_1 &= \mathbf{I}_1 \widetilde{\mathbf{N}}_1 \mathbf{I}_1 + \mathbf{I}_1 \widetilde{\mathbf{N}}_2 \mathbf{I}_2 + \mathbf{I}_2 \widetilde{\mathbf{N}}_3 \mathbf{I}_1 + \mathbf{I}_2 \widetilde{\mathbf{N}}_1^{\mathrm{T}} \mathbf{I}_2, \\ \mathbf{N}_2 &= \mathbf{I}_1 \widetilde{\mathbf{N}}_1 \mathbf{I}_2 + \mathbf{I}_1 \widetilde{\mathbf{N}}_2 \mathbf{I}_1 + \mathbf{I}_2 \widetilde{\mathbf{N}}_3 \mathbf{I}_2 + \mathbf{I}_2 \widetilde{\mathbf{N}}_1^{\mathrm{T}} \mathbf{I}_1, \\ \mathbf{N}_3 &= \mathbf{I}_2 \widetilde{\mathbf{N}}_1 \mathbf{I}_1 + \mathbf{I}_2 \widetilde{\mathbf{N}}_2 \mathbf{I}_2 + \mathbf{I}_1 \widetilde{\mathbf{N}}_3 \mathbf{I}_1 + \mathbf{I}_1 \widetilde{\mathbf{N}}_1^{\mathrm{T}} \mathbf{I}_2. \end{aligned} \tag{7.3b}$$

8. Conclusions

Two Stroh-like complex variable formalisms for the coupled stretching–bending analysis of composite laminates are presented in this paper. One is displacement formalism, and the other is mixed formalism. The former was introduced previously by the other researchers and re-derived in this paper by a more Stroh-like way, while the latter is established here to compensate the displacement formalism. In these two Stroh-like formalisms, the general solutions for the basic equations of lamination theory and their associated eigenrelations are all obtained in complex matrix form. From the results and discussions presented in this paper, we see that both the displacement and mixed formalisms are not perfectly alike to the Stroh formalism for two-dimensional problems. The displacement formalism is alike in general solution, whereas the mixed formalism is alike in eigenrelation. To combine the merits from both formalisms, we may use the general solutions formed by the displacement formalism and when there is a need to count on the eigenrelation we may use the eigenrelation from the mixed formalism.

By using the presently developed mixed formalism, the explicit expressions for the fundamental matrix and eigenvectors are obtained first time for the most general composite laminates with coupling. For both formalisms, almost all the relations have been purposely arranged to have the same form as those of the Stroh formalism for two-dimensional formalism. Due to the resemblance, almost all the formulae and mathematical techniques developed for two-dimensional problems can be transferred to the problems of composite laminates with in-plane and plate bending coupling. By simple analogy, many problems that cannot be solved previously, now have the possibility to be solved analytically.

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